

DISCONTINUITY OF STRAIGHTENING IN ANTIHOLOMORPHIC DYNAMICS

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ABSTRACT. Multicorns are the connectedness loci of unicritical anti-holomorphic polynomials. Numerical experiments show that ‘baby multicorns’ appear in multicorns, and in parameter spaces of various other families of rational maps. The principal aim of this article is to explain this ‘universality’ property of the multicorns: on the one hand, we give a precise meaning to the notion of ‘baby multicorns’, and on the other hand, we show that the dynamically natural straightening map from a ‘baby multicorn’, either in multicorns of even degree or in the real cubic locus, to the original multicorn is discontinuous at infinitely many explicit parameters. This is the first known example where straightening maps fail to be continuous on a real two-dimensional slice of a holomorphic family of holomorphic polynomials. The proof of discontinuity of straightening maps is carried out by showing that all non-real *umbilical cords* of the multicorns wiggle, which settles a conjecture made by various people including Hubbard, Milnor, and Schleicher.

We also prove some rigidity theorems for polynomial parabolic germs, which state that one can recover unicritical holomorphic and antiholomorphic polynomials from their parabolic germs.

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1. INTRODUCTION

Renormalization is one of the most powerful tools in the study of dynamical systems. In the celebrated paper [DH85], Douady and Hubbard developed the theory of polynomial-like maps to study renormalizations of complex polynomials, and proved the *straightening theorem* that allows one to study a sufficiently large iterate of a polynomial by associating a simpler dynamical system, namely a polynomial of smaller degree, to it. They used it to explain the existence of small homeomorphic copies of the Mandelbrot set in itself. The fact that baby Mandelbrot sets are homeomorphic to the original one, is in some sense, a strictly ‘no interaction amongst critical orbits’ phenomenon. The existence of critical orbit interactions for higher degree polynomials allow for much more complicated dynamical configurations, and the corresponding straightening maps are typically not as well-behaved as in the unicritical case.

Substantial progress in understanding the combinatorics and topology of straightening maps for higher degree polynomials has been made by Epstein (manuscript), the first author and Kiwi [IK12, Ino09] in recent years. While the situation in quadratic dynamics is extremely satisfactory where each baby Mandelbrot set is homeomorphic to the original one (even quasiconformally equivalent in some cases [Lyu99]), such a miracle cannot be expected in the parameter spaces of higher degree polynomials. The first author showed that straightening maps are typically discontinuous in the presence of critical orbit relations. The proof of discontinuity of straightening maps given in [Ino09], however, makes essential use of two complex dimensional bifurcations, and can not be applied to one-parameter families. Thus, the question whether straightening maps could fail to be continuous in one-parameter families, remained open. In particular, it was conjectured that straightening maps for quadratic antiholomorphic polynomials (which form a real two-dimensional slice of biquadratic polynomials) are discontinuous, provided that the renormalization period is odd. The main purpose of this paper is to prove this conjecture in complete generality. In fact, we prove discontinuity of straightening maps in every even degree unicritical antiholomorphic polynomial family.

The dynamics of quadratic antiholomorphic polynomials and its connectedness locus, the tricorn, was first studied in [CHRSC89], and their numerical experiments showed major structural differences between the Mandelbrot set and the tricorn; in particular, they observed that there are bifurcations from the period 1 hyperbolic component to period 2 hyperbolic components along arcs in the tricorn, in contrast to the fact that bifurcations are always attached at a single point in the Mandelbrot set. The bifurcation structure in the family of quadratic antiholomorphic polynomials was studied in [Win90]. However, it was Milnor who first observed the importance of the multicorns (which are the connectedness loci of unicritical antiholomorphic polynomials $\bar{z}^d + c$); he found little tricorn and multicorn-like sets as prototypical objects in the parameter space of real cubic polynomials [Mil92], and in the real slices of rational maps with two critical points [Mil00]. Nakane [Nak93] proved that the tricorn is connected, in analogy to Douady and Hubbard's classical proof on the Mandelbrot set. This generalizes naturally to multicorns of any degree. Later, Nakane and Schleicher, in [NS03], studied the structure of hyperbolic components of the multicorns via the multiplier map (even period case), and the critical value map (odd period case). These maps are branched coverings over the unit disk of degree $d - 1$ and $d + 1$ respectively, branched only over the origin. Hubbard and Schleicher [HS14] proved that the multicorns are not pathwise connected, confirming a conjecture of Milnor. Recently, in an attempt to explore the topological aspects of the parameter spaces of unicritical antiholomorphic polynomials, the combinatorics of external dynamical rays of such maps were studied in [Muk15b] in terms of orbit portraits, and this was used in [MNS15] where the bifurcation phenomena, boundaries of odd period hyperbolic components, and the combinatorics of parameter rays were described. The authors showed in [IM16] that many parameter rays of the multicorns non-trivially accumulate on persistently parabolic regions.

The multicorns can be thought of as objects of intermediate complexity between one dimensional, and higher dimensional parameter spaces. Douady's famous 'plough in the dynamical plane, and harvest in the parameter plane' principle continues to stand us in good stead since our parameter space is still real two-dimensional. But since the parameter dependence of anti-polynomials is only real-analytic, one cannot typically use complex analytic techniques to study the multicorns directly. This can be circumvented by passing to the second iterate, and embedding the family $\bar{z}^d + c$ in the family $\mathcal{F}_d = \{(z^d + a)^d + b : a, b \in \mathbb{C}\}$ of holomorphic polynomials. Thus the multicorn \mathcal{M}_d^* is the intersection of the real 2-dimensional slice $\{a = \bar{b}\}$ with the connectedness locus of \mathcal{F}_d , and hence it reflects several properties of higher dimensional parameter spaces. In fact, we will heavily exploit the critical orbit interactions of the polynomials $(z^d + a)^d + b$, and the existence of non-trivial deformation classes of parabolic parameters in the multicorns in our proof of discontinuity of straightening maps.

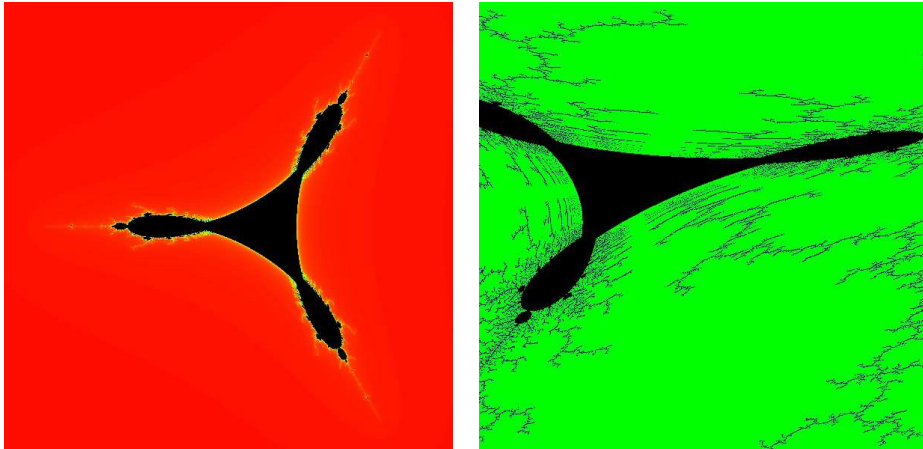


FIGURE 1. Left: The tricorn. Right: Wiggling of an umbilical cord on the root parabolic arc of a hyperbolic component of period 5 of the tricorn.

The combinatorics and topology of the multicorns differ in many ways from those of their holomorphic counterparts, the multibrot sets, which are the connectedness loci of degree d unicritical polynomials. At the level of combinatorics, this is manifested in the structure of orbit portraits [Muk15b, Theorem 2.6, Theorem 3.1]. The topological features of the multicorns have quite a few properties in common with the connectedness locus of real cubic polynomials, e.g. discontinuity of landing points of dynamical rays, bifurcation along arcs, existence of real-analytic curves containing q.c.-conjugate parabolic parameters, lack of local connectedness of the connectedness loci, non-landing stretching rays etc. [Lav89], [KN04], [HS14, Corollary 3.7], [IM16], [MNS15, Theorem 3.2, Theorem 6.2], [Muk15a]. These are in stark contrast with the multibrot sets.

Numerical experiments suggest that every odd period hyperbolic component of the multicorns is the basis of a small ‘copy’ of the multicorn itself, much like the Mandelbrot set. While it is true that an antiholomorphic analogue of the straightening theorem does provide us with a map from every small multicorn-like set to the original multicorn, it had been conjectured by various people, including Milnor, Hubbard, and Schleicher, that this map is discontinuous [HS14, MP12]. The first author recently gave a computer-assisted proof of this fact for a particular candidate [Ino14]. The principal goal of this paper is to prove this conjecture for every multicorn-like set contained in multicorns of even degree.

Theorem 1.1 (Discontinuity of Straightening, I). *Let d be even, c_0 be the center of a hyperbolic component H of odd period (other than 1) of \mathcal{M}_d^* ,*

and $\mathcal{R}(c_0)$ be the corresponding c_0 -renormalization locus. Then the straightening map $\chi_{c_0} : \mathcal{R}(c_0) \rightarrow \mathcal{M}_d^*$ is discontinuous (at infinitely many explicit parameters).

The proof of discontinuity is carried out by showing that the straightening map from a baby multicorn-like set to the original multicorn sends certain ‘wiggly’ curves to landing curves. More precisely, for even degree multicorns, there exist hyperbolic components H intersecting the real line, and their ‘umbilical cords’ land on the root parabolic arc on ∂H . In other words, such a component can be connected to the period 1 hyperbolic component by a path. However, we will prove that if H does not intersect the real line or its rotates, then no path γ contained in $\mathcal{M}_d^* \setminus \overline{H}$ can land on the root parabolic arc on ∂H (this holds for multicorns of any degree). The non-existence of such a path will be referred to as the ‘wiggling’ of ‘umbilical cords’ of non-real hyperbolic components. Hence, for even degree multicorns, the (inverse of the) straightening map sends a piece of the real line to a ‘wiggly’ curve; which is an obstruction to continuity.

The following theorem generalizes the main result of [HS14], and shows that path-connectivity fails to hold in a very strong sense for the multicorns. We should mention that this is a major topological difference from the Mandelbrot set. In fact, any two Yoccoz parameters (i.e. at most finitely renormalizable parameters) in the Mandelbrot set can be connected by an arc in the Mandelbrot set [Sch99, Theorem 5.6], [PR08].

Theorem 1.2 (Umbilical Cord Wiggling). *Let H be a hyperbolic component of odd period k of \mathcal{M}_d^* , \mathcal{C} be the root arc on ∂H , and \tilde{c} be the critical Ecalle height 0 parameter on \mathcal{C} . If there is a path $p : [0, \delta] \rightarrow \mathbb{C}$ with $p(0) = \tilde{c}$, and $p((0, \delta]) \subset \mathcal{M}_d^* \setminus \overline{H}$, then d is even, and $\tilde{c} \in \mathbb{R} \cup \omega \mathbb{R} \cup \omega^2 \mathbb{R} \cup \dots \cup \omega^d \mathbb{R}$, where $\omega = \exp(\frac{2\pi i}{d+1})$.*

The techniques used to prove Theorem 1.1 and Theorem 1.2 can be applied to study the tricorn-like sets in the parameter space of other polynomial families as well. The existence of small tricorn-like sets in the parameter plane of real cubic polynomials was numerically observed by Milnor [Mil92]. Let us illustrate how quadratic anti-polynomial-like behavior can be observed in the dynamical plane of a real cubic polynomial. Let $p(z) = -z^3 - 3a_0^2 z + b_0$, $a_0, b_0 \in \mathbb{R}$, $a_0 \geq 0$ be a post-critically finite real cubic polynomial with a bitransitive mapping scheme (see [MP12] for the definition of mapping schemes). The two critical points $c_1 := ia_0$ and $c_2 := -ia_0$ of p are complex conjugate, and have complex conjugate forward orbits. The assumption on the mapping scheme implies that there exists an $n \in \mathbb{N}$ such that $p^{\circ n}(c_1) = c_2$, $p^{\circ n}(c_2) = c_1$ (compare Figure 2). Suppose U is a neighborhood of the closure of the Fatou component containing c_1 such that $p^{\circ 2n} : U \rightarrow p^{\circ 2n}(U)$ is polynomial-like of degree 4. Then we have, $\iota(\overline{U}) \subset p^{\circ n}(U)$ (where \overline{U} is the topological closure of U , and ι is the complex conjugation map), i.e. $\overline{U} \subset (\iota \circ p^{\circ n})(U)$. Therefore $\iota \circ p^{\circ n} : U \rightarrow (\iota \circ p^{\circ n})(U)$

is a proper antiholomorphic map of degree 2, hence an anti-polynomial-like map of degree 2 (with a connected filled-in Julia set) defined on U . An antiholomorphic version of the straightening Theorem (Theorem 5.1) now yields a quadratic antiholomorphic map (with a connected filled-in Julia set) that is hybrid equivalent to $(\iota \circ p^{\circ n})|_U$. One can continue to perform this renormalization procedure as the real cubic polynomial p moves in the parameter space, and this defines a map from a suitable region in the parameter plane of real cubic polynomials to the tricorn. We will define these tricorn-like sets rigorously as a suitable renormalization locus $\mathcal{R}(a_0, b_0)$, and will define the dynamically natural ‘straightening map’ from $\mathcal{R}(a_0, b_0)$ to the tricorn \mathcal{M}_2^* . Using techniques similar to the ones described above, we will show that the non-real umbilical cords for these tricorn-like sets wiggle, and will conclude that:

Theorem 1.3 (Discontinuity of Straightening, II). *The straightening map $\chi_{a_0, b_0} : \mathcal{R}(a_0, b_0) \rightarrow \mathcal{M}_2^*$ is discontinuous (at infinitely many explicit parameters).*

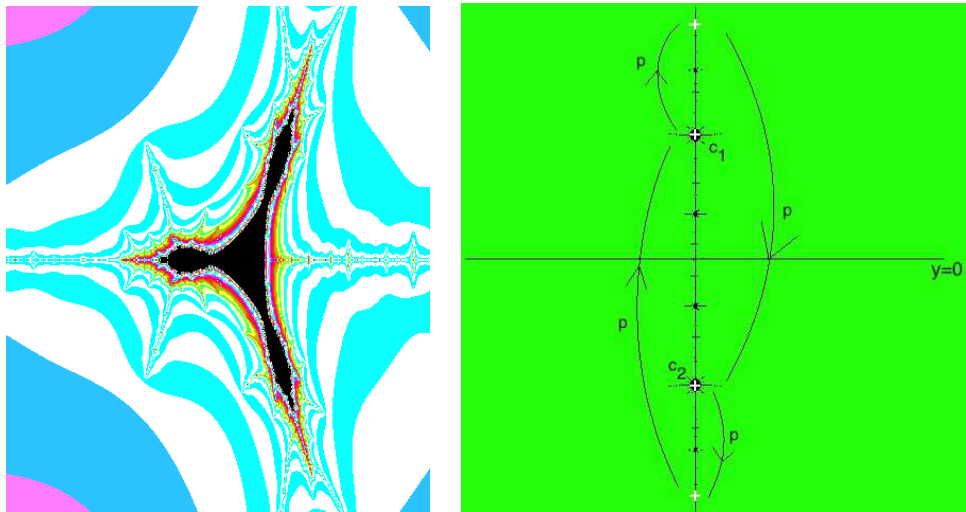


FIGURE 2. Left: A tricorn-like set in the parameter plane of real cubic polynomials. Right: The dynamical plane of a real cubic polynomial with real-symmetric critical orbits and a bitransitive mapping scheme.

The existence of non-landing umbilical cords for the multicorns was first proved by Hubbard and Schleicher [HS14] (see also [NS96]) under a strong assumption of non-renormalizability. The main technical tool in their proof is the theory of perturbation of antiholomorphic parabolic points [HS14, §4] [IM16, §2]. Using these perturbation techniques, they showed that the landing of an umbilical cord at \tilde{c} implies that a loose parabolic tree of $f_{\tilde{c}}$

would contain a real-analytic arc connecting two bounded Fatou components. With the assumption of non-renormalizability, one can deduce from the above statement that the entire parabolic tree is a real-analytic arc, and this implies that $f_{\tilde{c}}$ and $f_{\tilde{c}^*}$ (here, and in the sequel, z^* will stand for the complex conjugate of the complex number z) are conformally conjugate, proving that \tilde{c} lies on the real line (or one of its rotates). In order to demonstrate discontinuity of straightening maps, we need to get rid of the non-renormalizability hypothesis; i.e. we need to prove wiggling of umbilical cords for all non-real hyperbolic components. Evidently, in the general case, we have to adopt a different strategy which will be outlined soon.

The discontinuity of straightening maps discussed above is a rather topological phenomenon in the sense that straightening maps fail to be continuous because they do not preserve the ‘topology’ of connectedness loci. There is yet another, in fact a more conformal reason for straightening maps to be discontinuous which we describe now. In the presence of more than one critical points, for instance when two critical points are attracted by a single parabolic cycle, one can associate at least two different conformal conjugacy invariants with the parabolic cycle: i) the difference of the Fatou coordinates of two critical values (or their orbit representatives) in the attracting Ecalle cylinder, which we would call the *Fatou vector*, and ii) the holomorphic fixed point index of the parabolic cycle. The first invariant is an ‘internal’ conformal invariant of the dynamics, and is preserved by straightening maps. But the second one, which is an ‘external’ conformal invariant, is in general not preserved by straightening maps (since a hybrid equivalence does not necessarily preserve the external class of a polynomial-like map). Moreover, the Fatou vector can be quasi-conformally deformed giving rise to an analytic family of q.c. equivalent parabolic maps [Muk15c, §3]. Heuristically speaking, if straightening maps were continuous, they would preserve the geometry of the parameter space. In fact, we prove that continuity of straightening maps would force the above two conformal invariants to be uniformly related along every parabolic arc, and this is a very strong geometric condition that is almost too good to hold. Although we do not know how to rule this out in general, we do show that the relation between Fatou vector and parabolic fixed point index is not uniform for certain low period parabolic arcs of the tricorn. This shows that straightening maps between certain tricorn-like sets fail to be continuous essentially because it fails to preserve the ‘geometry’ of connectedness loci. In fact, our methods suggest that any two “copies” of the connectedness locus of biquadratic polynomials in the parameter space of cubic polynomials (compare [IK12]) are dynamically distinct; i.e. they are not homeomorphic via straightening maps.

One of the key steps in our proof of discontinuity of straightening maps is to extend a carefully constructed local conjugacy between parabolic germs to a semi-local (polynomial-like map) conjugacy, which allows us to conclude

that the corresponding polynomials are affinely conjugate. In general, we believe that two polynomial parabolic germs can be conformally conjugate only if the two polynomials are polynomial semi-conjugates of a common polynomial. Motivated by these considerations, we prove some rigidity principles for unicritical holomorphic and antiholomorphic polynomials with parabolic cycles. Let $\mathcal{M}_d^{\text{par}}$ be the set of parabolic parameters of the multibrot set \mathcal{M}_d . We prove the following theorem.

Theorem 1.4 (Parabolic Germs Determine Roots and Co-roots of Multibrot Sets). *For $i = 1, 2$, let $c_i \in \mathcal{M}_{d_i}^{\text{par}}$, z_i be the characteristic parabolic point of $p_{c_i}(z) = z^{d_i} + c_i$, and k_i be the period of z_i under p_{c_i} . If the restrictions $p_{c_1}^{\circ k_1}|_{N_{z_1}}$ and $p_{c_2}^{\circ k_2}|_{N_{z_2}}$ (where N_{z_i} is a sufficiently small neighborhood of z_i) are conformally conjugate, then $d_1 = d_2$, and p_{c_1} and p_{c_2} are affinely conjugate.*

Let us now elaborate on the organization of the paper. In Section 2, we will survey some known results about antiholomorphic dynamics, and the global combinatorial and topological structure of the multicorns. As mentioned earlier, using the implosion techniques as developed in [HS14, IM16], one can show that the landing of an umbilical cord at \tilde{c} implies that a loose parabolic tree of $f_{\tilde{c}}$ would contain a real-analytic arc connecting two bounded Fatou components. From the existence of a small real-analytic arc connecting two bounded Fatou components, we will show in Section 3 that the characteristic parabolic germs of $f_{\tilde{c}}$ and $f_{\tilde{c}^*}$ are conformally conjugate by a local biholomorphism that preserves the critical orbit tails. This is a fundamental step in our proof. Since there exists an infinite-dimensional family of conformal conjugacy classes of parabolic germs [Eca75, Vor81]; heuristically speaking, it is extremely unlikely that the parabolic germs of two conformally different polynomials would be conformally conjugate. The next step in our proof involves extending the local analytic conjugacy between parabolic germs to larger domains, step by step. In Section 4, we first extend this local conjugacy to the entire characteristic Fatou component, and then continue it to a neighborhood of the closure of the characteristic Fatou component. This gives us a pair of conformally conjugate polynomial-like restrictions, and applying a theorem of [Ino11], we conclude that some iterates of $f_{\tilde{c}}$ and $f_{\tilde{c}^*}$ are globally conjugate by a finite-to-finite holomorphic correspondence. This means that some iterates of $f_{\tilde{c}}$ and $f_{\tilde{c}^*}$ are (globally) polynomially semi-conjugate to a common polynomial. The final step in the proof of Theorem 1.2 is to conclude that \tilde{c} is conformally conjugate to a real parameter, by using the theory of decompositions of polynomials with respect to composition, which is due to Ritt [Rit22] and Engstrom [Eng41]. In Section 5, we will recall some general combinatorial and topological facts about straightening maps. Section 6 deals with a continuity property of straightening maps. In particular, we show that straightening maps induce homeomorphisms between the closures of odd period hyperbolic components both in the multicorns, and in the tricorn-like sets in the real cubic locus

(this has been independently proved in [BBM]). Subsequently, in Section 7, we will use the wiggling behavior of non-real umbilical cords to give a proof of Theorem 1.1. In Section 8, we prove Theorem 1.3, which asserts that the straightening map from any tricorn-like set in the real cubic locus to the original tricorn is discontinuous. In Section 9, we state a conjecture on a stronger (and more geometric) form of discontinuity of straightening maps to the effect that the baby multicorns are dynamically different from each other. We also provide positive evidence supporting the conjecture by demonstrating that the original tricorn is ‘dynamically’ distinct from the period 3 baby tricorn. We conclude the paper with an analysis of some local-global questions for polynomial parabolic germs. These questions are motivated by our proof of discontinuity of straightening maps, and are of independent interest in the theory of parabolic points. In Section 10, we first recall some known facts about extended horn maps, and give a proof of Theorem 1.4 using the mapping properties of extended horn maps. We also show that one can recover the parabolic parameters of the multicorns, up to some natural rotational and reflection symmetries, from their parabolic germs. In Section 11, we prove that the parabolic germ of a unicritical holomorphic polynomial (respectively anti-polynomial) is conformally conjugate to a real-symmetric parabolic germ if and only if the polynomial (respectively anti-polynomial) commutes with a global antiholomorphic involution (whose axis of symmetry passes through the parabolic point).

It is worth mentioning that the proof of discontinuity of straightening maps for general polynomial families given in [Ino09] also involves proving the existence of analytically conjugate polynomial-like restrictions. The main difference is that, in higher dimensional parameter spaces, continuity of straightening maps allows one to find richer perturbations to obtain analytically conjugate polynomial-like maps. Indeed, one of the main technical steps in [Ino09] is to show (using parabolic implosion techniques) that continuity of straightening maps forces certain hybrid equivalences to preserve the moduli of multipliers of repelling periodic points of certain polynomial-like maps, and this implies that the hybrid equivalence can be promoted to an analytic equivalence. On the other hand, the present proof employs a one-dimensional parabolic perturbation to first obtain an analytic conjugacy between parabolic germs, which is then promoted to an analytic conjugacy between polynomial-like restrictions. However, both the proofs have a common philosophy: that is to show that continuity of straightening maps would force certain hybrid equivalences to preserve some of the ‘external conformal information’.

To conclude, we should remark that more generally, one expects the existence of multicorn-like sets in any family of polynomials or rational maps with (at least) two critical orbits such that a pair of critical orbits are symmetric with respect to an antiholomorphic involution. Evidences of this fact can be found in the recent works on the parameter spaces of certain families of rational maps, such as the family of antipode preserving cubic rationals

[BBM15, BBM], Blaschke products [CFG15], etc. Although not all of our techniques can be applied to such families of rational maps, the parabolic perturbation arguments, and the local consequence of umbilical cord landing (Section 3) do work in a general setting, and paves the way for studying analogous questions for rational maps.

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2. ANTIHOLOMORPHIC DYNAMICS, AND GLOBAL STRUCTURE OF THE MULTICORNS

In this section, we briefly recall some known results on antiholomorphic dynamics, and their parameter spaces, which we will have need for in the rest of the paper.

2.1. Basic Definitions. Any unicritical antiholomorphic polynomial, after an affine change of coordinates, can be written in the form $f_c(z) = \bar{z}^d + c$ for some $d \geq 2$, and $c \in \mathbb{C}$. In analogy to the holomorphic case, the set of all points which remain bounded under all iterations of f_c is called the *Filled-in Julia set* $K(f_c)$. The boundary of the Filled-in Julia set is defined to be the *Julia set* $J(f_c)$, and the complement of the Julia set is defined to be its *Fatou set* $F(f_c)$. This leads, as in the holomorphic case, to the notion of *connectedness locus* of degree d unicritical antiholomorphic polynomials:

Definition. The *multicorn* of degree d is defined as $\mathcal{M}_d^* = \{c \in \mathbb{C} : K(f_c) \text{ is connected}\}$. The multicorn of degree 2 is called the *tricorn*.

The basin of infinity, and the corresponding Böttcher coordinate play a vital role in the dynamics of polynomials. In the antiholomorphic setting, we need a parallel notion of Böttcher coordinates. By [Nak93, Lemma 1], there is a conformal map φ_c near ∞ that conjugates f_c to \bar{z}^d . As in the holomorphic case, φ_c extends conformally to an equipotential containing c , when $c \notin \mathcal{M}_d^*$, and extends as a biholomorphism from $\hat{\mathbb{C}} \setminus K(f_c)$ onto $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ when $c \in \mathcal{M}_d^*$.

Definition (Dynamical Ray). The dynamical ray $R_c(\vartheta)$ of f_c at an angle ϑ is defined as the pre-image of the radial line at angle ϑ under φ_c .

The dynamical ray $R_c(\vartheta)$ maps to the dynamical ray $R_c(-d\vartheta)$ under f_c . We refer the readers to [NS03, Section 3], [Muk15b] for details on the combinatorics of the landing pattern of dynamical rays for unicritical antiholomorphic polynomials. The next result was proved by Nakane [Nak93].

Theorem 2.1 (Real-Analytic Uniformization). *The map $\Phi : \mathbb{C} \setminus \mathcal{M}_d^* \rightarrow \mathbb{C} \setminus \overline{\mathbb{D}}$, defined by $c \mapsto \varphi_c(c)$ (where φ_c is the Böttcher coordinate near ∞*

for f_c) is a real-analytic diffeomorphism. In particular, the multicorns are connected.

The previous theorem also allows us to define parameter rays of the multicorns.

Definition (Parameter Ray). The parameter ray at angle ϑ of the multicorn \mathcal{M}_d^* , denoted by \mathcal{R}_ϑ^d , is defined as $\{\Phi^{-1}(re^{2\pi i\vartheta}) : r > 1\}$, where Φ is the real-analytic diffeomorphism from the exterior of \mathcal{M}_d^* to the exterior of the closed unit disc in the complex plane constructed in Theorem 2.1.

Remark. Some comments should be made on the definition of the parameter rays. Observe that unlike the multibrot sets, the parameter rays of the tricorns are not defined in terms of the Riemann map of the exterior. In fact, the Riemann map of the exterior of \mathcal{M}_d^* has no obvious dynamical meaning. We have defined the parameter rays via a dynamically defined diffeomorphism of the exterior of \mathcal{M}_d^* , and it is easy to check that this definition of parameter rays agrees with the notion of stretching rays (which are dynamically defined objects) in the family of polynomials $(z^d + a)^d + b$.

Let, $\omega = \exp(\frac{2\pi i}{d+1})$. The antiholomorphic polynomials f_c and $f_{\omega c}$ are conformally conjugate via the linear map $z \mapsto \omega z$. It follows that:

Lemma 2.2 (Symmetry). *Let, $\omega = \exp(\frac{2\pi i}{d+1})$. Then, $f_c \sim f_{\omega c} \sim f_{\omega^2 c} \sim \dots \sim f_{\omega^d c}$. In particular, \mathcal{M}_d^* has a $(d+1)$ -fold rotational symmetry.*

2.2. Hyperbolic Components of Odd Periods, and Bifurcations.

One of the main features of the antiholomorphic parameter spaces is the existence of abundant parabolics. In particular, the boundaries of odd period hyperbolic components of the tricorns consist only of parabolic parameters.

Lemma 2.3 (Indifferent Dynamics of Odd Period). *The boundary of a hyperbolic component of odd period k consists entirely of parameters having a parabolic orbit of exact period k . In local conformal coordinates, the $2k$ -th iterate of such a map has the form $z \mapsto z + z^{q+1} + \dots$ with $q \in \{1, 2\}$.*

Proof. See [MNS15, Lemma 2.5]. □

This leads to the following classification of odd periodic parabolic points.

Definition (Parabolic Cusps). A parameter c will be called a *cusp point* if it has a parabolic periodic point of odd period such that $q = 2$ in the previous lemma. Otherwise, it is called a *simple* parabolic parameter.

In holomorphic dynamics, the local dynamics in attracting petals of parabolic periodic points is well-understood: there is a local coordinate ψ which conjugates the first-return dynamics to translation by $+1$ in a right half plane (see Milnor [Mil06, Section 10]). Such a coordinate ψ is called a *Fatou coordinate*. Thus the quotient of the petal by the dynamics is isomorphic to a bi-infinite cylinder, called the *Ecalte cylinder*. Note that Fatou coordinates are uniquely determined up to addition by a complex constant.

In antiholomorphic dynamics, the situation is at the same time restricted and richer. Indifferent dynamics of odd period is always parabolic because for an indifferent periodic point of odd period k , the $2k$ -th iterate is holomorphic with positive real multiplier, hence parabolic as described above. On the other hand, additional structure is given by the antiholomorphic intermediate iterate.

Lemma 2.4 (Fatou Coordinates). *Suppose z_0 is a parabolic periodic point of odd period k of f_c with only one petal (i.e. c is not a cusp), and U is a periodic Fatou component with $z_0 \in \partial U$. Then there is an open subset $V \subset U$ with $z_0 \in \partial V$, and $f_c^{\circ k}(V) \subset V$ so that for every $z \in U$, there is an $n \in \mathbb{N}$ with $f_c^{\circ nk}(z) \in V$. Moreover, there is a univalent map $\psi: V \rightarrow \mathbb{C}$ with $\psi(f_c^{\circ k}(z)) = \overline{\psi(z)} + 1/2$, and $\psi(V)$ contains a right half plane. This map ψ is unique up to horizontal translation.*

Proof. See [HS14, Lemma 2.3]. \square

The map ψ will be called an *antiholomorphic Fatou coordinate* for the petal V . The antiholomorphic iterate interchanges both ends of the Ecalle cylinder, so it must fix one horizontal line around this cylinder (the *equator*). The change of coordinate has been so chosen that the equator is the projection of the real axis. We will call the vertical Fatou coordinate the *Ecalle height*. Its origin is the equator. Of course, the same can be done in the repelling petal as well. We will refer to the equator in the attracting (respectively repelling) petal as the attracting (respectively repelling) equator. The existence of this distinguished real line, or equivalently an intrinsic meaning to Ecalle height, is specific to antiholomorphic maps.

The Ecalle height of the critical value plays a special role in antiholomorphic dynamics. The next theorem proves the existence of real-analytic arcs of non-cusp parabolic parameters on the boundaries of odd period hyperbolic components of the tricorn.

Theorem 2.5 (Parabolic Arcs). *Let \tilde{c} be a parameter such that $f_{\tilde{c}}$ has a parabolic orbit of odd period, and suppose that \tilde{c} is not a cusp. Then \tilde{c} is on a parabolic arc in the following sense: there exists a real-analytic arc of non-cusp parabolic parameters $c(t)$ (for $t \in \mathbb{R}$) with quasiconformally equivalent but conformally distinct dynamics of which \tilde{c} is an interior point, and the Ecalle height of the critical value of $f_{c(t)}$ is t .*

Proof. See [MNS15, Theorem 3.2]. \square

For an isolated fixed point $\hat{z} = f(\hat{z})$ where $f: U \rightarrow \mathbb{C}$ is a holomorphic function on a connected open set $U \subset \mathbb{C}$ containing \hat{z} , the residue fixed point index of f at \hat{z} is defined to be the complex number

$$\iota(f, \hat{z}) = \frac{1}{2\pi i} \oint \frac{dz}{z - f(z)}.$$

where we integrate in a small loop in the positive direction around \hat{z} . If the multiplier $\rho := f'(\hat{z})$ is not equal to $+1$, then a simple computation shows that $\iota(f, \hat{z}) = 1/(1 - \rho)$. If z_0 is a parabolic fixed point with multiplier $+1$, then in local holomorphic coordinates the map can be written as $f(w) = w + w^{q+1} + \alpha w^{2q+1} + \dots$ (putting $\hat{z} = 0$), and α is a conformal invariant (in fact, it is the unique formal invariant other than q : there is a formal, not necessarily convergent, power series that formally conjugates f to the first three terms of its series expansion). A simple calculation shows that α equals the parabolic fixed point index. The ‘résidu itératif’ of f at the parabolic fixed point \hat{z} of multiplier 1 is defined as $\left(\frac{q+1}{2} - \alpha\right)$. It is easy to see that the fixed point index does not depend on the choice of complex coordinates, and is a conformal invariant (compare [Mil06, §12]).

The typical structure of the boundaries of hyperbolic components of even periods is that there are $d - 1$ isolated root or co-root points which are connected by curves off the root locus. For hyperbolic components of odd periods, the story is in a certain sense just the opposite: the analogues of roots or co-roots are now arcs, of which there are $d + 1$, and the analogues of the connecting curves are the isolated cusp points between the arcs. There is trouble, of course, where components of even and odd periods meet, and we get bifurcations along arcs: the root of the even period component stretches along parts of two arcs. This phenomenon was first observed in [CHRSC89] for the main component of the tricorn. The precise statement is given in the following results, which were proved in [HS14, Proposition 3.7, Theorem 3.8, Corollary 3.9]. The proof of this fact uses the concept of holomorphic fixed point index. The main idea is that when several simple fixed points merge into one parabolic point, each of their indices tends to ∞ , but the sum of the indices tends to the index of the resulting parabolic fixed point, which is finite.

Lemma 2.6 (Fixed Point Index on Parabolic Arc). *Along any parabolic arc of odd period, the fixed point index is a real valued real-analytic function that tends to $+\infty$ at both ends.*

Theorem 2.7 (Bifurcations Along Arcs). *Every parabolic arc of period k intersects the boundary of a hyperbolic component of period $2k$ at the set of points where the fixed-point index is at least 1, except possibly at (necessarily isolated) points where the index has an isolated local maximum with value 1. In particular, every parabolic arc has, at both ends, an interval of positive length at which bifurcation from a hyperbolic component of odd period k to a hyperbolic component of period $2k$ occurs.*

Now let H be a hyperbolic component of odd period k , and \mathcal{C} be a parabolic arc on the boundary of H . The previous theorem tells that there is an even period hyperbolic component H' (of period $2k$) bifurcating from H across \mathcal{C} . Furthermore, the corresponding bifurcation locus $\mathcal{C} \cap \partial H \cap \partial H'$ (from H to H' across \mathcal{C}) is precisely the set of parameters on \mathcal{C} where

the fixed-point index is at least 1, except possibly the (necessarily isolated) points where the index has an isolated local maximum with value 1. Hence, $\mathcal{C} \cap \partial H \cap \partial H'$ is the union of a sub-arc of \mathcal{C} and possibly finitely many isolated points on \mathcal{C} . In our next lemma, we will slightly sharpen the statement of Theorem 2.7 by ruling out any such “accidental intersection” of ∂H and $\partial H'$. More precisely, we will show that $\mathcal{C} \cap \partial H \cap \partial H'$ contains no isolated point; i.e. it is a sub-arc of \mathcal{C} .

Let $c : \mathbb{R} \rightarrow \mathcal{C}$ be the critical Ecalle height parametrization of \mathcal{C} (see Theorem 2.5). By [IM16, Corollary 5.2], there is no bifurcation across the Ecalle height 0 parameter $c(0)$; i.e. $c(0) \notin \partial H'$. Therefore, $\mathcal{C} \cap \partial H \cap \partial H'$ is contained either in $c(0, +\infty)$ or in $c(-\infty, 0)$. We can assume without loss of generality that $\mathcal{C} \cap \partial H \cap \partial H' \subset c(0, +\infty)$. We define

$$h_0 := \inf\{h > 0 : c(h, +\infty) \subset \partial H \cap \partial H'\},$$

and

$$\tilde{h}_0 := \inf\{h > 0 : c(h) \in \partial H \cap \partial H'\}.$$

Clearly, $0 < \tilde{h}_0 \leq h_0$. Observe that if $\mathcal{C} \cap \partial H \cap \partial H'$ contains an isolated point, then h_0 would be strictly greater than \tilde{h}_0 .

Lemma 2.8 (No Accidental Bifurcation). $h_0 = \tilde{h}_0$. Consequently, $\mathcal{C} \cap \partial H \cap \partial H'$ contains no isolated point.

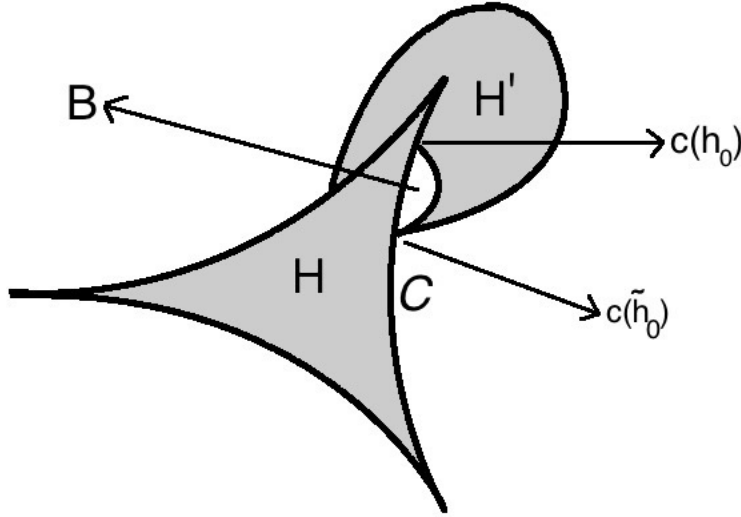


FIGURE 3. The hyperbolic components H and H' are shown in grey. If $\partial H \cap \partial H'$ contains an isolated point, then $\overline{H} \cup \overline{H'}$ will have a bounded complementary component B . Therefore, all parameters on $(\overline{B} \cap \partial H') \setminus \mathcal{C}$ will be irrationally indifferent.

Proof. Let us assume that $h_0 > \tilde{h}_0$. So $\mathbb{C} \setminus (\overline{H} \cup \overline{H'})$ has a bounded component B . Note that $\overline{B} \setminus \{c(\tilde{h}_0)\}$ is contained in the interior of \mathcal{M}_d^* (compare Figure 3), and hence cannot contain a parabolic parameter of even period since every parabolic parameter of even period is the landing point of some external parameter ray of \mathcal{M}_d^* (compare [MNS15, Lemma 7.4]). Therefore every point of $(\overline{B} \cap \partial H') \setminus \mathcal{C}$ must have an irrationally indifferent $2k$ -periodic cycle, and their multipliers depend continuously on the parameter. By continuity and connectedness of \mathbb{R}/\mathbb{Z} , this multiplier map must be constant on $(\overline{B} \cap \partial H') \setminus \mathcal{C}$. This means that there is a ϑ in $(\mathbb{R} \setminus \mathbb{Q})/\mathbb{Z}$ such that each parameter c on $(\overline{B} \cap \partial H') \setminus \mathcal{C}$ has a $2k$ -periodic cycle of multiplier $e^{2\pi i \vartheta}$. This is impossible because the set of parameters c in \mathcal{M}_d^* such that f_c has a *non-repelling* periodic orbit of given period $2k$ and given multiplier μ (where $|\mu| \leq 1$) is finite.¹ This contradiction proves that $h_0 = \tilde{h}_0$. \square

Following [MNS15], we classify parabolic arcs into two types.

Definition (Root Arcs and Co-Root Arcs). We call a parabolic arc a *root arc* if, in the dynamics of any parameter on this arc, the parabolic orbit disconnects the Julia set. Otherwise, we call it a *co-root arc*.

Definition (Characteristic Parabolic Point). Let f_c have a parabolic periodic point. The characteristic parabolic point of f_c is the unique parabolic point on the boundary of the critical value Fatou component of f_c .

Definition (Rational Lamination). The rational lamination of a holomorphic or antiholomorphic polynomial f (with connected Julia set) is defined as an equivalence relation on \mathbb{Q}/\mathbb{Z} such that $\vartheta_1 \sim \vartheta_2$ if and only if the dynamical rays $R(\vartheta_1)$ and $R(\vartheta_2)$ land at the same point of $J(f)$. The rational lamination of f is denoted by $\lambda(f)$.

The structure of the hyperbolic components of odd period plays an important role in the global topology of the parameter spaces. Let H be a hyperbolic component of odd period $k \neq 1$ (with center c_0) of the multicorn \mathcal{M}_d^* . The first return map of the closure of the characteristic Fatou component of c_0 fixes exactly $d + 1$ points on its boundary. Only one of these fixed points disconnects the Julia set, and is the landing point of two distinct dynamical rays at $2k$ -periodic angles. Let the set of the angles of these two rays be $S' = \{\alpha_1, \alpha_2\}$. Each of the remaining d fixed points is the landing point of precisely one dynamical ray at a k -periodic angle; let the collection of the angles of these rays be $S = \{\vartheta_1, \vartheta_2, \dots, \vartheta_d\}$. We can,

¹For a proof of this result in the case of attracting and rationally indifferent cycles, see [MNS15, Lemma 2.7]. The case of irrationally indifferent cycles follows from the arguments of [MNS15, Lemma 2.7] combined with [BCL⁺15, Theorem 1.1] which allows one to associate a distinct critical orbit to each non-repelling cycle of a polynomial. In either case, the result is a consequence of Bézout's theorem since the set of the parameters under consideration is bounded, and is contained in the intersection of a pair of algebraic curves.

possibly after renumbering, assume that $0 < \alpha_1 < \vartheta_1 < \vartheta_2 < \cdots < \vartheta_d < \alpha_2$ and $\alpha_2 - \alpha_1 < \frac{1}{d}$.

By [MNS15, Theorem 1.2], ∂H is a simple closed curve consisting of $d+1$ parabolic arcs, and the same number of cusp points such that every arc has two cusp points at its ends. Exactly one of these $d+1$ parabolic arcs is a root arc, and the parameter rays at angles α_1 and α_2 accumulate on this arc. The characteristic parabolic point in the dynamical plane of any parameter on this root arc is the landing point of precisely two dynamical rays at angles α_1 and α_2 . The rest of the d parabolic arcs on ∂H are co-root arcs. Each of these co-root arcs contains the accumulation set of exactly one parameter ray at an angle ϑ_i , and such that the characteristic parabolic point in the dynamical plane of any parameter on this co-root arc is the landing point of precisely one dynamical ray at angle ϑ_i . Furthermore, the rational lamination remains constant throughout the closure of the hyperbolic component H except at the cusp points.

By [NS03, Theorem 5.6], every even period hyperbolic component H' of \mathcal{M}_d^* is homeomorphic to \mathbb{D} , and the corresponding multiplier map $\mu_{H'} : H' \rightarrow \mathbb{D}$ is a real-analytic $(d-1)$ -fold branched cover ramified only over the origin.

Definition (Internal Rays of Even Period Components). An *internal ray* of an even period hyperbolic component H' of \mathcal{M}_d^* is an arc $\gamma \subset H$ starting at the center such that there is an angle ϑ with $\mu_{H'}(\gamma) = \{re^{2\pi i\vartheta} : r \in [0, 1)\}$.

Remark. Since $\mu_{H'}$ is a $(d-1)$ -to-one map, an internal ray of H' with a given angle is not uniquely defined. In fact, a hyperbolic component has $(d-1)$ internal rays with any given angle ϑ .

Let us record the following basic property of internal rays of even period hyperbolic components. Although a proof of this fact has never appeared before, we believe that the result is somewhat folklore.

Lemma 2.9 (Landing of Internal Rays for Even Period Components). *For every hyperbolic component H' of even period, all internal rays land. The landing point of an internal parameter ray at angle ϑ has an indifferent periodic orbit with multiplier $e^{2\pi i\vartheta}$. If the period of this orbit is odd, then $\vartheta = 0$, and the landing point is a parabolic cusp.*

Proof. Let $2k$ be the period of the hyperbolic component H' . Every parameter c in the accumulation set of an internal ray at angle ϑ has an indifferent periodic point z satisfying $f_c^{\circ 2k}(z) = z$, $(f_c^{\circ 2k})'(z) = e^{2\pi i\vartheta}$. So an internal ray lands whenever such boundary parameters are isolated. If H' does not bifurcate from an odd period hyperbolic component, then indifferent parameters of a given multiplier are isolated on $\partial H'$ (recall that the set of parameters c in \mathcal{M}_d^* such that f_c has a *non-repelling* periodic orbit of given period $2k$ and given multiplier μ , where $|\mu| \leq 1$, is finite). On the other hand, if H' bifurcates from an odd period hyperbolic component, then indifferent parameters of multiplier $e^{2\pi i\vartheta}$ may be non-isolated on $\partial H'$ only if

$\vartheta = 0$, and the parabolic orbit of the corresponding parameters have *odd period* k . Therefore we only need to consider this one exceptional case. In all other cases, the candidate accumulation set of an internal ray is discrete, and hence the ray must land.

Let \mathcal{R} be an internal ray at angle 0 of H' (bifurcating from an odd period component H of period k), and c be an accumulation point of \mathcal{R} such that f_c has a k -periodic parabolic cycle of multiplier 1. We claim that c must be a cusp point of period k on ∂H . Since cusp points of a given period are isolated [MNS15, Lemma 2.9], it would follow that \mathcal{R} lands at a cusp point, completing the proof of the lemma. To prove the claim, let us choose a sequence $\{c_n\} \subset \mathcal{R}$ with $c_n \rightarrow c$ such that each of the two $2k$ -periodic attracting cycles of $f_{c_n}^{\circ 2}$ has multiplier λ_n (in general, these two attracting cycles have complex conjugate multipliers, but since \mathcal{R} is an internal ray at angle 0, the multipliers are real, and hence equal). If c has a simple parabolic cycle, then the residue fixed point index of the parabolic cycle of $f_c^{\circ 2}$ is equal to

$$\lim_{n \rightarrow \infty} \frac{1}{1 - \lambda_n} + \frac{1}{1 - \lambda_n} = \lim_{n \rightarrow \infty} \frac{2}{1 - \lambda_n} = +\infty,$$

which is impossible. Therefore, c must be a cusp point of period k on ∂H . \square

Remark. It follows from the above lemma that if H' is an even period hyperbolic component (of period $2k$) bifurcating from an odd period component H (of period k), then no internal ray of H' accumulates on the parabolic arcs of H .

The above lemma has an interesting corollary. We will use the terminologies of Lemma 2.8. For any h in \mathbb{R} , let us denote the residue fixed point index of the unique parabolic cycle of $f_{c(h)}^{\circ 2}$ by $\text{ind}_{\mathcal{C}}(f_{c(h)}^{\circ 2})$. By Lemma 2.8, we can assume without loss of generality that the set of parameters on \mathcal{C} across which bifurcation from H to H' occurs is precisely $c[h_0, +\infty)$; i.e. $\mathcal{C} \cap \partial H \cap \partial H' = c[h_0, +\infty)$.

Corollary 2.10. *The function*

$$\begin{aligned} \text{ind}_{\mathcal{C}} : \mathbb{R} &\rightarrow \mathbb{R} \\ h &\mapsto \text{ind}_{\mathcal{C}}(f_{c(h)}^{\circ 2}). \end{aligned}$$

is strictly increasing for $h \geq h_0$.

Proof. Pick $h \geq h_0$. It follows from the proof of Lemma 2.9 that an internal ray \mathcal{R} at angle 0 of H' lands at the cusp point $\lim_{h \rightarrow +\infty} c(h)$, and the impression of \mathcal{R} contains the sub-arc $c[h_0, +\infty)$ (of the parabolic arc \mathcal{C}). So we can choose a sequence $(H' \ni) c_n \rightarrow c(h) \in \mathcal{C}$ such that $\mu_{H'}(c_n) = r_n e^{2\pi i \vartheta_n} = x_n + i y_n$, where $r_n \uparrow 1$, and $\vartheta_n \rightarrow 0$. It follows that

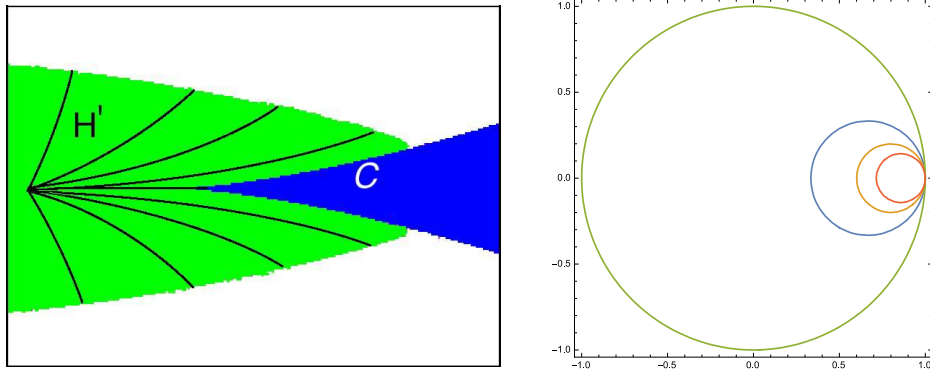


FIGURE 4. Left: A schematic picture of internal rays of an even period hyperbolic component bifurcating from an odd period hyperbolic component. Right: The circles C_h (in the uniformizing plane) centered at $(1 - \frac{1}{\tau_h})$ with radius $\frac{1}{\tau_h}$ for various values of τ_h . All of these circles touch at 1. As τ_h increases, the circles get nested. As $c_n \rightarrow c(h)$, $\mu_{H'}(c_n)$ converges to 1 asymptotically to the circle C_h .

$$\begin{aligned} \text{ind}_{\mathcal{C}}(f_{c(h)}^{\circ 2}) &= \lim_{n \rightarrow \infty} \left(\frac{1}{1 - r_n e^{2\pi i \vartheta_n}} + \frac{1}{1 - r_n e^{-2\pi i \vartheta_n}} \right) \\ &= \lim_{n \rightarrow \infty} \left(\frac{2(1 - x_n)}{(1 - x_n)^2 + y_n^2} \right) \end{aligned}$$

Set $\tau_h := \text{ind}_{\mathcal{C}}(f_{c(h)}^{\circ 2})$. The above relation implies that as n tends to infinity, $\mu_{H'}(c_n)$ tends to 1 asymptotically to the circle $C_h :=$

$$\frac{2(1 - x)}{(1 - x)^2 + y^2} = \tau_h$$

$$\text{i.e., } \left(x - \left(1 - \frac{1}{\tau_h} \right) \right)^2 + y^2 = \frac{1}{\tau_h^2}.$$

For any fixed $\vartheta \neq 0$ and sufficiently close to 0, parameters (on \mathcal{C}) with larger critical Ecalle height h are approximated by parameters c (in H') with smaller values of $r := |\mu_{H'}(c)|$ (compare Figure 4(Left)). But it is easy to see either by direct computation or from Figure 4(Right) that a fixed radial line at angle $\vartheta \neq 0$ intersects circles C_h with smaller radii (i.e. larger τ_h) at points with smaller values of r (by the nesting pattern of the circles). The upshot of this analysis is that larger values of h correspond to larger (strictly speaking, no smaller) values of τ_h . This implies that the function $\text{ind}_{\mathcal{C}}$ is non-decreasing on $[h_0, +\infty)$. Since $\text{ind}_{\mathcal{C}}$ is a non-constant real-analytic function, it must be strictly increasing on $[h_0, +\infty)$. \square

It should be mentioned that unlike for the multibrot sets, not every (external) parameter ray of the multicorns land at a single point.

Theorem 2.11 (Non-Landing Parameter Rays). *The accumulation set of every parameter ray accumulating on the boundary of a hyperbolic component of odd period (except period one) of \mathcal{M}_d^* contains an arc of positive length.*

See [IM16, Theorem 1.1, Theorem 4.2] for a detailed account on non-landing parameter rays, and for a complete classification of which rays land, and which ones do not.

2.3. Parabolic Tree. We will need the concept of parabolic trees, which are defined in analogy with Hubbard trees for post-critically finite polynomials. Our definition will follow [HS14, Section 5]. The proofs of the basic properties of the tree can be found in [Sch00, Lemma 3.5, Lemma 3.6].

Definition (Parabolic Tree). If c lies on a parabolic root arc of odd period k , we define a *loose parabolic tree* of f_c as a minimal tree within the filled-in Julia set that connects the parabolic orbit and the critical orbit, so that it intersects the critical value Fatou component along a simple f_c^{ck} -invariant curve connecting the critical value to the characteristic parabolic point, and it intersects any other Fatou component along a simple curve that is an iterated pre-image of the curve in the critical value Fatou component. Since the filled-in Julia set of a parabolic polynomial is locally connected, and hence path connected, any loose parabolic tree connecting the parabolic orbit is uniquely defined up to homotopies within bounded Fatou components. It is easy to see that any loose parabolic tree intersects the Julia set in a Cantor set, and these points of intersection are the same for any loose tree (note that for simple parabolics, any two periodic Fatou components have disjoint closures).

By construction, the forward image of a loose parabolic tree is again a loose parabolic tree. A simple standard argument (analogous to the post-critically finite case) shows that the boundary of the critical value Fatou component intersects the tree at exactly one point (the characteristic parabolic point), and the boundary of any other bounded Fatou component meets the tree in at most d points, which are iterated pre-images of the characteristic parabolic point [Sch00, Lemma 3.5] [EMS16, Lemma 3.2, Lemma 3.3]. The critical value is an endpoint of any loose parabolic tree. All branch points of a loose parabolic tree are either in bounded Fatou components or repelling (pre-)periodic points; in particular, no parabolic point (of odd period) is a branch point.

3. UMBILICAL CORD LANDING, AND CONFORMAL CONJUGACY OF PARABOLIC GERMS

A standing convention: In the rest of the paper, we will denote the complex conjugate of a complex number z either by \bar{z} or by z^* . The complex

conjugation map will be denoted by ι , i.e. $\iota(z) = z^*$. The image of a set U under complex conjugation will be denoted as $\iota(U)$, and the topological closure of U will be denoted by \overline{U} .

Our goal in this section is to apply the perturbation techniques developed in [HS14, §4] [IM16, §2] to prove a local consequence of umbilical cord landing. We will work with a fixed hyperbolic component H of odd period $k \neq 1$. Let \mathcal{C} be the root arc on ∂H , \tilde{c} be the Ecalle height 0 parameter on \mathcal{C} , and z_1 be the characteristic parabolic point of $f_{\tilde{c}}$. Assume further that the dynamical rays $R_{\tilde{c}}(\vartheta)$ and $R_{\tilde{c}}(\vartheta')$ land at the characteristic parabolic point z_1 . By symmetry, there is a hyperbolic component $\iota(H)$ (which is just the reflection of H with respect to the real line) of the same period k such that \tilde{c}^* is the Ecalle height 0 parameter on the root arc $\iota(\mathcal{C})$ of $\iota(H)$. The characteristic parabolic point of $f_{\tilde{c}^*}$ is z_1^* .

We begin with an elementary lemma.

Lemma 3.1. *Any two bounded Fatou components of $f_{\tilde{c}}$ have disjoint closures.*

Proof. Let, U_1 and U_2 be two distinct Fatou components of $f_{\tilde{c}}$ with $\overline{U_1} \cap \overline{U_2} \neq \emptyset$. By taking iterated forward images, we can assume that U_1 and U_2 are periodic. Then the intersection consists of a repelling periodic point x of some period n .

It is easy to see the first return map of $\overline{U_1}$ (and of $\overline{U_2}$) fixes x , and x disconnects the Julia set; hence, x is the ‘root’ of U_1 (as well as of U_2). It follows from [NS03, Corollary 4.2] that $n = k$. But this contradicts the fact that every periodic Fatou component of $f_{\tilde{c}}$ has exactly one root, and there is exactly one cycle of periodic bounded Fatou components (for instance, use the fact that on the root parabolic arc on ∂H , the attracting periodic points of both U_1 and U_2 would merge with the root x yielding a double parabolic point!). This contradiction proves the lemma. \square

The following lemma essentially states that landing of an umbilical cord at \tilde{c} implies a (local) regularity property of a loose parabolic tree of $f_{\tilde{c}}$. Although the proof can be extracted from [HS14, Lemma 5.10, Theorem 6.1], we prefer working out the details here as the organization of the present paper differs from that of [HS14].

Lemma 3.2. *If there is a path $p : [0, \delta] \rightarrow \mathbb{C}$ with $p(0) = \tilde{c}$, and $p((0, \delta]) \subset \mathcal{M}_d^* \setminus \overline{H}$, then the repelling equator at z_1 is contained in a loose parabolic tree of $f_{\tilde{c}}$.*

Proof. Since any two bounded Fatou components of $f_{\tilde{c}}$ have disjoint closures, and the inverse images of the characteristic parabolic point z_1 are dense on the Julia set, it follows that any parabolic tree must traverse infinitely many bounded Fatou components, and intersect their boundaries at pre-parabolic points. Furthermore, any loose parabolic tree intersects the Julia set at a Cantor set of points. We first claim that the part of any loose parabolic tree contained in the repelling petal of z_1 intersects the Julia set entirely

along the repelling equator. To do this, we will assume the contrary, and will arrive at a contradiction.

If the part of the parabolic tree contained in the repelling petal were not contained in the equator, then there would be a point w_0 (say, repelling pre-periodic) in the intersection having Ecalle height $h > 0$. We construct a sequence (w_n) so that $w_{n+1} := f_{\tilde{c}}^{-\circ 2k}(w_n)$ choosing a local branch of $f_{\tilde{c}}^{-\circ 2k}$ that fixes z_1 , and so that all w_n are in the repelling petal of z_1 . Therefore, $w_n \rightarrow z_1$ as $n \rightarrow \infty$, and all w_n have the same Ecalle height h . Similarly, let $w'_n := f_{\tilde{c}}^{\circ k}(w_n)$, then $w'_n \rightarrow z_1$, and all these points have Ecalle heights $-h$. As w_0 is on the parabolic tree, which is invariant, it follows that w_0 is accessible from outside of the filled-in Julia set on both sides of the tree, so each w_n and w'_n is the landing point of (at least) two dynamic rays, ‘above’ and ‘below’ the tree. If ϑ_n be the angle of dynamical ray landing at w_n from below, then it follows that $\vartheta_n \rightarrow \vartheta$ (say) as $n \rightarrow \infty$, and $R_{\tilde{c}}(\vartheta_n)$ traverses (at least) the interval $[-h/2, h/2]$ of (outgoing) Ecalle heights. Analogously, if ϑ'_n be the angle of dynamical ray landing at w'_n from above, then it follows that $\vartheta'_n \rightarrow \vartheta'$ as $n \rightarrow \infty$, and $R_{\tilde{c}}(\vartheta'_n)$ traverses (at least) the interval $[-h/2, h/2]$ of (outgoing) Ecalle heights.

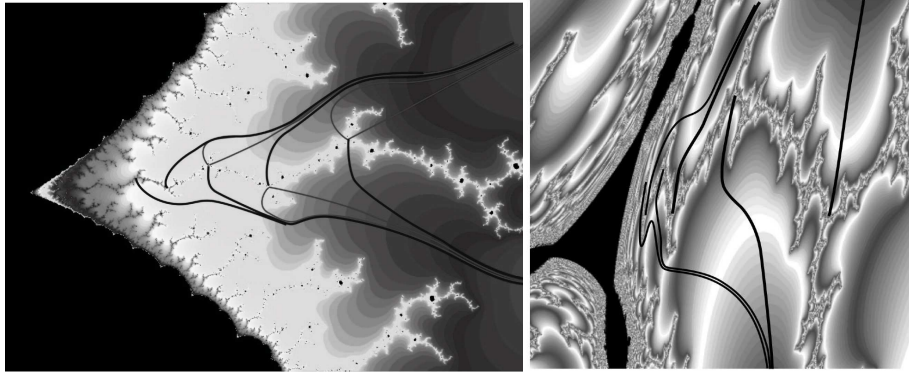


FIGURE 5. Left: Dynamical rays crossing the repelling equator in the dynamical plane. Right: The corresponding parameter rays obstructing the existence of the required path p . (Figures courtesy Dierk Schleicher.)

The dynamical rays at angles ϑ_n and ϑ'_n , and their landing points depend equicontinuously (i.e. the uniform continuous dependence on the parameter is independent of n) on the parameter (as they are pre-periodic rays). The projection of these rays onto the outgoing Ecalle cylinder is also continuous. Hence, there exists a neighborhood U of \tilde{c} in the parameter space such that if $c' \in \overline{U} \setminus H$, then the projection of the dynamical rays $R_{c'}(\vartheta_n)$ and $R_{c'}(\vartheta'_n)$ onto the outgoing cylinder of $f_{c'}^{\circ 2k}$ traverse the interval of (outgoing) Ecalle heights $[-h/3, h/3]$.

By assumption, there is a path $p : [0, \delta] \rightarrow \mathbb{C}$ with $p(0) = \tilde{c}$, and $p((0, \delta]) \subset \mathcal{M}_d^* \setminus \overline{H}$. By choosing a smaller δ , we can assume that $p((0, \delta]) \subset \overline{U} \setminus \overline{H}$. For $s > 0$, the critical orbit of $f_{p(s)}$ “transits” from the incoming Ecalle cylinder to the outgoing cylinder; as $s \downarrow 0$, the image of the critical orbit in the outgoing Ecalle cylinder has (outgoing) Ecalle height tending to 0, while the phase tends to infinity [IM16, Lemma 2.5]. Therefore, there is $s \in (0, \delta)$ arbitrarily close to 0 for which the critical value, projected into the incoming cylinder, and sent by the transfer map to the outgoing cylinder, lands on the projection of the dynamical rays $R_{p(s)}(\vartheta_n)$ (or $R_{p(s)}(\vartheta'_n)$). But in the dynamics of $f_{p(s)}$, this means that the critical value is in the basin of infinity, i.e. such a parameter $p(s)$ lies outside \mathcal{M}_d^* . This contradicts our assumption that $p((0, \delta]) \subset \mathcal{M}_d^* \setminus \overline{H}$, and proves that the part of any loose parabolic tree contained in the repelling petal of z_1 must intersect the Julia set entirely along the repelling equator.

In fact, the above argument essentially shows that the existence of any dynamical ray (in the repelling petal at z_1) traversing an interval of outgoing Ecalle heights $[-x, x]$ with $x > 0$ would destroy the existence of the required path p (compare Figure 5). In other words, for the existence of such a path p , no dynamical ray should ‘cross’ the repelling equator. Therefore, the repelling equator is contained in the filled-in Julia set $K(f_{\tilde{c}})$; i.e. the repelling equator forms the part of a loose parabolic tree. \square

So far, we have more or less proceeded in the same direction as in [HS14]. More precisely, we have showed that in order that an umbilical cord lands, the corresponding antiholomorphic polynomial must have a loose parabolic tree whose intersection with the repelling petal is an analytic arc (observe that the equator is an analytic arc; i.e. the image of the real line under a biholomorphism). But without any assumption on non-renormalizability, we *cannot* conclude anything about the global structure of the parabolic tree. To circumvent this problem, we will adopt a different ‘local to global’ principle. The following lemma shows that the local regularity of the parabolic tree, established in the previous lemma, has a very surprising consequence on the corresponding parabolic germs.

Lemma 3.3 (Local Analytic Conjugacy of Parabolic Germs). *If the repelling equator of $f_{\tilde{c}}$ at z_1 is contained in a loose parabolic tree, then the parabolic germs given by the restrictions of $f_{\tilde{c}}^{\circ 2k}$ and $f_{\tilde{c}^*}^{\circ 2k}$ at their characteristic parabolic points are conformally conjugate by a local biholomorphism that maps $f_{\tilde{c}}^{\circ kr}(\tilde{c})$ to $f_{\tilde{c}^*}^{\circ kr}(\tilde{c}^*)$, for r large enough.*

Proof. Pick any bounded Fatou component U (different from the characteristic Fatou component) that the repelling equator hits. Assume that the equator intersects ∂U at some pre-parabolic point z' . Consider a small piece Γ' of the equator with z' in its interior. Since z' eventually falls on the parabolic orbit, some large iterate of f_c maps z' to z_1 by a local biholomorphism

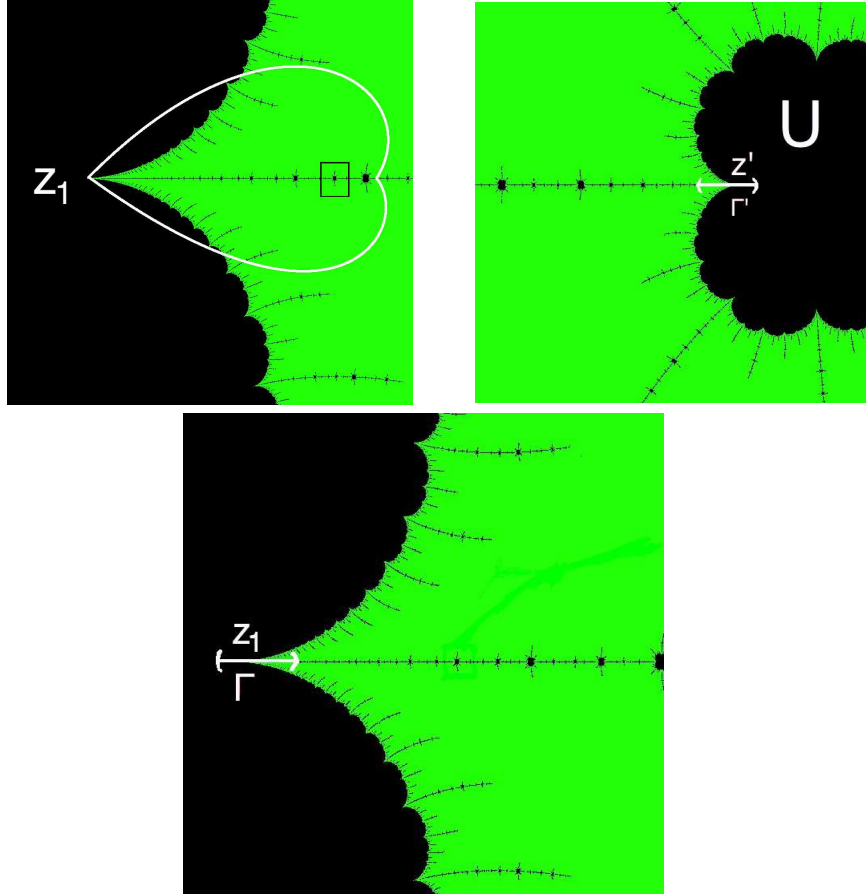


FIGURE 6. Top left: A repelling petal at the characteristic parabolic point is enclosed by the white curve. The repelling equator at the characteristic parabolic point is contained in the filled-in Julia set. The square box contains a Fatou component U such that z' is a point of intersection of ∂U , and the repelling equator. Top right: A blow-up of the Fatou component U contained in the box shown in the left figure. Bottom: The piece Γ' maps to an invariant analytic arc Γ which passes through the characteristic point z_1 , and lies in the filled-in Julia set.

carrying Γ' to an analytic arc Γ (say, $\Gamma = f_c^{\circ n}(\Gamma')$) passing through z_1 (compare Figure 6). We will show that Γ agrees with the repelling equator (up to truncation). Indeed, the repelling equator, and the curve Γ are both parts of two loose parabolic trees (any forward iterate of a loose parabolic tree is again a loose parabolic tree), and hence must coincide along a cantor set of points on the Julia set. As analytic arcs, they must thus coincide up to truncation. In particular, the part of Γ not contained in the characteristic

Fatou component is contained in the repelling equator, and is forward invariant. Straighten the analytic arc Γ to an interval $(-\varepsilon, \varepsilon) \subset \mathbb{R}$ by a local biholomorphism $\alpha : V \rightarrow \mathbb{C}$ such that $z_1 \in V$, and $\alpha(z_1) = 0$ (for convenience, we choose V such that it is symmetric with respect to Γ). This local biholomorphism conjugates the parabolic germ of $f_{\tilde{c}}^{\circ 2k}$ at z_1 to a germ that fixes 0. Moreover, the conjugated germ maps small enough positive reals to positive reals. Clearly, this must be a real germ. Thus, the parabolic germ of $f_{\tilde{c}}^{\circ 2k}$ at z_1 is analytically conjugate to a real germ.

Observe that $\iota : z \mapsto z^*$ is a topological conjugacy between $f_{\tilde{c}}$ and $f_{\tilde{c}^*}$. One can carry out the preceding construction with $f_{\tilde{c}^*}$, and show that the parabolic germ of $f_{\tilde{c}^*}^{\circ 2k}$ at z_1^* is analytically conjugate to a real germ. In fact, the role of Γ' is now played by $\iota(\Gamma')$, and hence, the role of Γ is played by $f_{\tilde{c}^*}^{\circ n}(\iota(\Gamma')) = \iota(\Gamma)$. Then the biholomorphism $\iota \circ \alpha \circ \iota : \iota(V) \rightarrow \mathbb{C}$ straightens $\iota(\Gamma)$. Conjugating the parabolic germ of $f_{\tilde{c}^*}^{\circ 2k}$ at z_1^* by $\iota \circ \alpha \circ \iota$, one recovers the same real germ as in the previous paragraph. Thus, the parabolic germs given by the restrictions of $f_{\tilde{c}}^{\circ 2k}$ and $f_{\tilde{c}^*}^{\circ 2k}$ at their characteristic parabolic points are analytically conjugate. Moreover, since the critical orbits of $f_{\tilde{c}}^{\circ 2k}$ lie on the equator, and since the equator is mapped to the real line by α , the conjugacy $\eta := (\iota \circ \alpha \circ \iota)^{-1} \circ \alpha$ preserves the critical orbits; i.e. it maps $f_{\tilde{c}}^{\circ kr}(\tilde{c})$ to $f_{\tilde{c}^*}^{\circ kr}(\tilde{c}^*)$ (for r large enough, so that $f_{\tilde{c}}^{\circ kr}(\tilde{c})$ is contained in the domain of definition of α). \square

Remark. It follows from the previous lemma that the real-analytic curve Γ passing through z_1 is invariant under $f_{\tilde{c}}^{\circ k}$. Indeed, Γ is formed by parts of the attracting equator, the repelling equator, and the parabolic point z_1 .

Remark. Observe that $f_{\tilde{c}}^{\circ 2k}$ has three critical points, and two (infinite) critical orbits in the characteristic Fatou component $U_{\tilde{c}}$. Two of these three critical points (of $f_{\tilde{c}}^{\circ 2k}|_{U_{\tilde{c}}}$) are mapped to the same point by $f_{\tilde{c}}^{\circ 2k}$ so that they lie on the same critical orbit of $f_{\tilde{c}}^{\circ 2k}$, and the third one lies on the other critical orbit of $f_{\tilde{c}}^{\circ 2k}$. Hence, the two critical orbits of $f_{\tilde{c}}^{\circ 2k}|_{U_{\tilde{c}}}$ are dynamically distinct.

We want to emphasize the fact that the conjugacy $\eta = (\iota \circ \alpha \circ \iota)^{-1} \circ \alpha$ constructed (from the condition that an umbilical cord lands at \tilde{c}) in the previous lemma is special: it maps (the tails of) each of the two (dynamically distinct) critical orbits of $f_{\tilde{c}}^{\circ 2k}$ to (the tails of) the corresponding critical orbit $f_{\tilde{c}^*}^{\circ 2k}$. In fact, the parabolic germs given by the restrictions of $f_{\tilde{c}}^{\circ 2k}$ and $f_{\tilde{c}^*}^{\circ 2k}$ at their characteristic parabolic points are *always* conformally conjugate by $\iota \circ f_{\tilde{c}}^{\circ k}$; but this local conjugacy exchanges the two post-critical orbits, which have different topological dynamics. Hence this local conjugacy has no chance of being extended to the entire parabolic basin.

3.1. A Brief Digression to Parabolic Germs. We have showed that if an umbilical cord lands at \tilde{c} , then the restriction of $f_{\tilde{c}}^{\circ 2k}$ at its characteristic parabolic point is analytically conjugate to a real germ. Let us denote the local reflection with respect to the curve Γ (as in the previous lemma) by ι_{Γ} .

Then, ι_Γ commutes with $f_{\tilde{c}}^{\circ k}$. Therefore, $f_{\tilde{c}}^{\circ 2k} = (f_{\tilde{c}}^{\circ k} \circ \iota_\Gamma) \circ (\iota_\Gamma \circ f_{\tilde{c}}^{\circ k}) = g^{\circ 2}$, where $g = f_{\tilde{c}}^{\circ k} \circ \iota_\Gamma = \iota_\Gamma \circ f_{\tilde{c}}^{\circ k}$. Thus, the parabolic germ given by the restriction of $f_{\tilde{c}}^{\circ 2k}$ in a neighborhood of z_1 is (locally) the second iterate of a holomorphic germ (which is a holomorphic germ with a parabolic fixed point at z_1 and multiplier 1).

By the classical theory of conformal conjugacy classes of parabolic germs, one knows that parabolic germs are extremely rigid. In fact, there is an infinite-dimensional family of conformally different parabolic germs [Eca75, Vor81]. With this in mind, the conclusion of Lemma 3.3, and the properties of the parabolic germ $f_{\tilde{c}}^{\circ 2k}$ at its characteristic parabolic point discussed in the previous paragraph, seem very unlikely to hold unless the polynomial $f_{\tilde{c}}^{\circ 2k}$ has a strong global symmetry. In fact, in the next section, we will prove that this can happen if and only if \tilde{c} lies on the real line or on one of it rotates. The proof, however, depends heavily on the structure of the polynomial $f_{\tilde{c}}^{\circ 2k}$. But it seems reasonable to try to understand the global implications of a local information about a polynomial parabolic germ, in more general settings. In particular, we ask the following questions:

Question 3.4 (From Germs to Polynomials).

- (1) *Let p be a complex polynomial with a parabolic fixed point at 0 with multiplier 1.*
 - (a) *If the parabolic germ $p|_{B(0,\varepsilon)}$ is locally conformally conjugate to a real parabolic germ, or*
 - (b) *if the parabolic germ $p|_{B(0,\varepsilon)}$ is locally the second iterate of a holomorphic germ,**can we conclude that the polynomial p has a corresponding global property?*
- (2) *Let p_1 and p_2 be two complex polynomials with parabolic fixed points at 0 with multiplier 1. If the two parabolic germs p_1 and p_2 at the origin are conformally conjugate, are p_1 and p_2 globally semi-conjugate or semi-conjugate from/to a common polynomial?*

Conditions (1a) and (1b) on the parabolic germ of p translate into corresponding conditions on its extended (lifted) horn map (such that its domain is maximal), i.e. they are real symmetric, or they commute with translation by $1/2$. This implies that the domains of the extended lifted horn maps are real-symmetric, or invariant under translation by $1/2$. Condition (2) implies that the horn maps of p_1 and p_2 at 0 are in fact the same, up to pre and post composition by multiplications with non-zero complex numbers.

For an explicit polynomial p , it may be possible to check whether the restriction of p in a neighborhood of 0 is locally the second iterate of a holomorphic germ, by looking at the corresponding horn map at 0, say h^- . All one needs to check for this is whether h^- is of the form

$$\sum_{n \in \mathbb{N}} a_n w^{2n-1}, \quad a_n \in \mathbb{C}.$$

Indeed, let ψ^{att} and ψ^{rep} be the attracting and repelling Fatou coordinates for p at 0; then these Fatou coordinates will conjugate the local compositional square root of p to the translation $z \mapsto z + 1/2$ (since a local square root commutes with p , it would induce a conformal isomorphism of the Ecalle cylinder whose second iterate is $z \mapsto z + 1$, and the only conformal isomorphism of \mathbb{C}/\mathbb{Z} with this property is $z \mapsto z + 1/2$). Hence, the lifted horn maps $\psi^{\text{att}} \circ (\psi^{\text{rep}})^{-1}$ would commute with $z \mapsto z + 1/2$, and the horn maps would commute with $w \mapsto -w$. Therefore, we would have $h^-(-w) = -h^-(w)$, and hence, h^- must be of the above form. In fact, this necessary condition on the horn map is also sufficient to ensure that the corresponding parabolic germ is locally a second iterate.

We will return to some of these local-global questions in Section 10 and Section 11.

4. EXTENDING THE LOCAL CONJUGACY

The main goal of this section is to prove that umbilical cords never land away from the real line or its rotates. More precisely, we will show that the existence of a path as in Lemma 3.2 would imply that $\tilde{c} \in \mathbb{R} \cup \omega\mathbb{R} \cup \omega^2\mathbb{R} \cup \dots \cup \omega^d\mathbb{R}$, where $\omega = e^{\frac{2\pi i}{d+1}}$. To this end, we will first extend the local analytic conjugacy between the parabolic germs, constructed in Lemma 3.3, to a conformal conjugacy between polynomial-like restrictions. This would allow us to apply a theorem of [Ino11] to deduce that the maps $f_{\tilde{c}}^{\circ 2k}$ and $f_{\tilde{c}^*}^{\circ 2k}$ are conjugate by an irreducible holomorphic correspondence. In other words, we will show that $f_{\tilde{c}}^{\circ 2k}$ and $f_{\tilde{c}^*}^{\circ 2k}$ are polynomially semi-conjugate to a common polynomial p . The final step involves proving that $f_{\tilde{c}}^{\circ 2k}$ and $f_{\tilde{c}^*}^{\circ 2k}$ are, in fact, affinely conjugate.

Recall that \tilde{c} is the Ecalle height 0 parameter on the root parabolic arc \mathcal{C} of the hyperbolic component H (period k), z_1 its characteristic parabolic point, and $U_{\tilde{c}}$ its characteristic Fatou component. We choose a Riemann map $\varphi_{\tilde{c}}$ of $U_{\tilde{c}}$ normalized so that it sends the critical value \tilde{c} to 0, and its homeomorphic extension to the boundary sends the parabolic point on $\partial U_{\tilde{c}}$ to 1. Then, $\varphi_{\tilde{c}}$ conjugates the first holomorphic return map $f_{\tilde{c}}^{\circ 2k}$ on $U_{\tilde{c}}$ to a Blaschke product B . Furthermore, the Ecalle height 0 condition implies that the images of the two critical orbits of $f_{\tilde{c}}^{\circ 2k}$ under an attracting Fatou coordinate are related by translation by $1/2$. It follows that the local compositional square root of $f_{\tilde{c}}^{\circ 2k}$ in an attracting petal (i.e. translation by $1/2$ pulled back by the Fatou coordinate) can be analytically extended throughout the immediate basin $U_{\tilde{c}}$. Therefore, the first return map $f_{\tilde{c}}^{\circ 2k}$ on $U_{\tilde{c}}$ is the second iterate of a holomorphic map g preserving $U_{\tilde{c}}$ with a parabolic fixed point of multiplier $+1$ at z_1 . Since $f_{\tilde{c}}^{\circ 2k}$ has two critical points in $U_{\tilde{c}}$, g is unicritical. Hence, the Riemann map $\varphi_{\tilde{c}}$ conjugates g to the Blaschke product $\tilde{B}(w) = \frac{(d+1)w^d + (d-1)}{(d+1) + (d-1)w^d}$. It follows that $B(w) = \tilde{B}^{\circ 2}(w)$. We record this fact in the following lemma.

Lemma 4.1 (Ecalte Height Zero Basins Are Conformally Conjugate). *The first holomorphic return map of an immediate basin of a parabolic point of odd period of a critical Ecalte height 0 parameter is conformally conjugate to $B(w) = \tilde{B}^{\circ 2}(w)$, where $\tilde{B}(w) = \frac{(d+1)w^d + (d-1)}{(d+1) + (d-1)w^d}$. In particular, they are conformally conjugate to each other.*

At this point, we know that $f_{\tilde{c}}^{\circ 2k}$ and $f_{\tilde{c}^*}^{\circ 2k}$, restricted to the characteristic Fatou components, are conformally conjugate, and the corresponding parabolic germs are also conformally conjugate by a ‘critical orbit’-preserving local biholomorphism. The lemma essentially shows that these two conjugacies can be glued together.

Lemma 4.2 (Extension to The Immediate Basin). *The conformal conjugacy η between the parabolic germs of $f_{\tilde{c}}^{\circ 2k}$ and $f_{\tilde{c}^*}^{\circ 2k}$ at their characteristic parabolic points can be extended to a conformal conjugation between the dynamics in the immediate basins.*

Proof. We need to choose our conformal change of coordinates symmetrically for $f_{\tilde{c}}$ and $f_{\tilde{c}^*}$.

Let us choose the attracting Fatou coordinate $\psi_{\tilde{c}}^{\text{att}}$ in $U_{\tilde{c}}$ normalized so that it maps the equator to the real line, and $\psi_{\tilde{c}}^{\text{att}}(\tilde{c}) = 0$. This naturally determines our preferred attracting Fatou coordinate $\psi_{\tilde{c}^*}^{\text{att}} := \iota \circ \psi_{\tilde{c}}^{\text{att}} \circ \iota$ for $f_{\tilde{c}^*}$ at its characteristic parabolic point z_1^* , and we have $\psi_{\tilde{c}^*}^{\text{att}}(\tilde{c}^*) = 0$.

Recall that we constructed a conformal conjugacy $\eta = \iota \circ \alpha^{-1} \circ \iota \circ \alpha : V \rightarrow \iota(V)$ between the parabolic germs $f_{\tilde{c}}^{\circ 2k}$ and $f_{\tilde{c}^*}^{\circ 2k}$ at their characteristic parabolic germs in Lemma 3.3. η maps some attracting petal (not necessarily containing \tilde{c}) $P \subset V$ of $f_{\tilde{c}}^{\circ 2k}$ at z_1 to some attracting petal $\iota(P) \subset \iota(V)$ of $f_{\tilde{c}^*}^{\circ 2k}$ at z_1^* . Hence, $\psi_{\tilde{c}}^{\text{att}} \circ \eta^{-1}$ is an attracting Fatou coordinate for $f_{\tilde{c}^*}^{\circ 2k}$ at z_1^* . By the uniqueness of Fatou coordinates, $\psi_{\tilde{c}}^{\text{att}} \circ \eta^{-1}(z) = \psi_{\tilde{c}^*}^{\text{att}}(z) + a$, for some $a \in \mathbb{C}$, and for all z in their common domain of definition. There is some large n for which $f_{\tilde{c}^*}^{\circ 2kn}(\tilde{c}^*)$ belongs to $\iota(V)$, the domain of definition of η^{-1} . By definition,

$$\begin{aligned}
 \psi_{\tilde{c}}^{\text{att}} \circ \eta^{-1}(f_{\tilde{c}^*}^{\circ 2kn}(\tilde{c}^*)) &= \psi_{\tilde{c}}^{\text{att}} \circ \alpha^{-1} \circ \iota \circ \alpha \circ \iota \circ f_{\tilde{c}^*}^{\circ 2kn}(\tilde{c}^*) \\
 &= \psi_{\tilde{c}}^{\text{att}} \circ \alpha^{-1} \circ \iota \circ \alpha \circ f_{\tilde{c}}^{\circ 2kn} \circ \iota(\tilde{c}^*) \\
 &= \psi_{\tilde{c}}^{\text{att}} \left(\alpha^{-1} \left(\iota \left(\alpha \left(f_{\tilde{c}}^{\circ 2kn}(\tilde{c}) \right) \right) \right) \right) \\
 &= \psi_{\tilde{c}}^{\text{att}} \left(\alpha^{-1} \left(\alpha \left(f_{\tilde{c}}^{\circ 2kn}(\tilde{c}) \right) \right) \right) \\
 &= \psi_{\tilde{c}}^{\text{att}} \left(f_{\tilde{c}}^{\circ 2kn}(\tilde{c}) \right) \\
 &= n .
 \end{aligned}$$

This holds since \tilde{c} lies on the equator, and α maps $f_{\tilde{c}}^{\circ 2nk}(\tilde{c})$ to the real line. But,

$$\begin{aligned} \psi_{\tilde{c}^*}^{\text{att}}(f_{\tilde{c}^*}^{\circ 2kn}(\tilde{c}^*)) &= \iota \circ \psi_{\tilde{c}}^{\text{att}} \circ \iota \left(\left(f_{\tilde{c}}^{\circ 2nk}(\tilde{c}) \right)^* \right) \\ &= n. \end{aligned}$$

This shows that $a = 0$, and hence, $\eta = (\psi_{\tilde{c}^*}^{\text{att}})^{-1} \circ \psi_{\tilde{c}}^{\text{att}}$ on P .

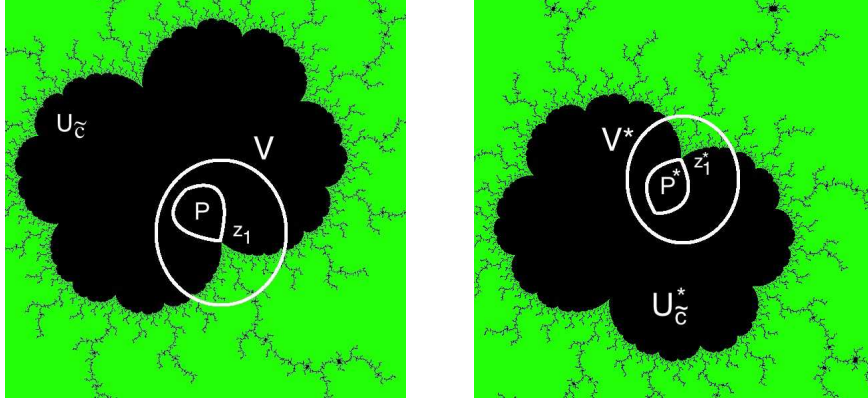


FIGURE 7. The germ conjugacy $\eta : V \rightarrow \iota(V)$, and the basin conjugacy $\chi : U_{\tilde{c}} \rightarrow \iota(U_{\tilde{c}})$ agree with $(\psi_{\tilde{c}^*}^{\text{att}})^{-1} \circ \psi_{\tilde{c}}^{\text{att}} : P \rightarrow \iota(P)$.

We also fix the Riemann map $\varphi_{\tilde{c}} : U_{\tilde{c}} \rightarrow \mathbb{D}$ which conjugates $f_{\tilde{c}}^{\circ 2k}$ on $U_{\tilde{c}}$ to the Blaschke product B in the previous lemma. Since the immediate basin of $f_{\tilde{c}^*}$ at its characteristic parabolic point z_1^* is $\iota(U_{\tilde{c}})$, $\iota \circ \varphi_{\tilde{c}} \circ \iota$ is the preferred Riemann map of the basin that sends the critical value \tilde{c}^* to 0, sends the parabolic point z_1^* to 1, and conjugates $f_{\tilde{c}^*}^{\circ 2k}$ to the Blaschke product B . A similar argument as above shows that the isomorphism $\chi := \iota \circ \varphi_{\tilde{c}}^{-1} \circ \iota \circ \varphi_{\tilde{c}} : U_{\tilde{c}} \rightarrow \iota(U_{\tilde{c}})$ agrees with $(\psi_{\tilde{c}^*}^{\text{att}})^{-1} \circ \psi_{\tilde{c}}^{\text{att}}$ on P , and hence extends the local conjugacy η to the entire immediate basin $U_{\tilde{c}}$ (compare Figure 7), such that it conjugates $f_{\tilde{c}}^{\circ 2k}$ on $U_{\tilde{c}}$ to $f_{\tilde{c}^*}^{\circ 2k}$ on $\iota(U_{\tilde{c}})$. \square

Abusing notations, let us denote the extended conjugacy from $U_{\tilde{c}} \cup V$ onto $\iota(U_{\tilde{c}} \cup V)$ of the previous lemma by η . Our next goal is to extend η to a neighborhood of $\overline{U_{\tilde{c}}}$ (the topological closure of $U_{\tilde{c}}$).

Lemma 4.3 (Extension to The Closure of The Basin). *η can be extended conformally to a neighborhood of $\overline{U_{\tilde{c}}}$.*

Proof. Observe that the basin boundaries are locally connected, and hence by Carathéodory's theorem, the conformal conjugacy η extends as a homeomorphism from $\partial U_{\tilde{c}}$ onto $\partial \iota(U_{\tilde{c}})$. Moreover, η extends analytically across the point z_1 . At this point, the existence of the required extension follows from [BE07, Lemma 2, Lemma 3]. However, we have a more straightforward

proof (essentially using the same idea) as our maps are unbranched on the Julia set.

By Montel's theorem, $\bigcup_n f_{\tilde{c}}^{\circ 2kn}(V \cap \partial U_{\tilde{c}}) = \partial U_{\tilde{c}}$. As none of the $f_{\tilde{c}}^{\circ 2kn}$ have critical points on $\partial U_{\tilde{c}}$, we can extend η in a neighborhood of each point of $\partial U_{\tilde{c}}$ by simply using the equation $\eta \circ f_{\tilde{c}}^{\circ 2kn} = f_{\tilde{c}^*}^{\circ 2kn} \circ \eta$. Since all of these extensions at various points of $\partial U_{\tilde{c}}$ extend the already defined (and conformal) common map η , the uniqueness of analytic continuations yields an analytic extension of η in a neighborhood of $\overline{U_{\tilde{c}}}$. By construction, this extension is clearly a proper holomorphic map, and assumes every point in $\iota(U_{\tilde{c}})$ precisely once. Therefore, the extended η from a neighborhood of $\overline{U_{\tilde{c}}}$ onto a neighborhood of $\iota(\overline{U_{\tilde{c}}})$ has degree 1, and is a conformal conjugacy between $f_{\tilde{c}}^{\circ 2k}$ and $f_{\tilde{c}^*}^{\circ 2k}$. \square

We are now ready to apply the ‘local to global’ result from [Ino11].

Lemma 4.4 (Global Semi-conjugacy). *There exist polynomials p , p_1 and p_2 such that $f_{\tilde{c}}^{\circ 2k} \circ p_1 = p_1 \circ p$, $f_{\tilde{c}^*}^{\circ 2k} \circ p_2 = p_2 \circ p$, and $\deg p_1 = \deg p_2$.*

Proof. Note that $f_{\tilde{c}}^{\circ 2k}$ (respectively $f_{\tilde{c}^*}^{\circ 2k}$) restricted to a small neighborhood of $\overline{U_{\tilde{c}}}$ (respectively $\iota(\overline{U_{\tilde{c}}})$) is polynomial-like of degree d^2 , and it follows from the previous lemma that these two polynomial-like maps are conformally conjugate. Applying Theorem [Ino11, Theorem 1] to this situation, we obtain the existence of polynomials p , p_1 , and p_2 such that the required semi-conjugacies hold. Since $f_{\tilde{c}}^{\circ 2k}$ and $f_{\tilde{c}^*}^{\circ 2k}$ are topologically conjugate by ι , it follows from the proof of Theorem [Ino11, Theorem 1] that $\deg p_1 = \deg p_2$ (observe that the product dynamics $(f_{\tilde{c}}^{\circ 2k}, f_{\tilde{c}^*}^{\circ 2k})$ is globally self-conjugate by $(\iota \times \iota) \circ q$, where $q : \mathbb{C}^2 \rightarrow \mathbb{C}^2$, $q(z, w) = (w, z)$). \square

In order to finish the proof of Theorem 1.2, we need to use a classification of semi-conjugate polynomials proved in [Ino11, Appendix A]. The results are based on the work of Ritt and Engstrom.

Let \mathcal{S} be the set of all affine conjugacy classes of triples (f, g, h) of polynomials of degree at least two such that $f \circ h = h \circ g$, where we say that two triples (f_1, g_1, h_1) and (f_2, g_2, h_2) are affinely conjugate if there exist affine maps σ_1, σ_2 such that

$$\begin{aligned} f_2 &= \sigma_1 \circ f_1 \circ \sigma_1^{-1}, \\ g_2 &= \sigma_2 \circ g_1 \circ \sigma_2^{-1}, \\ h_2 &= \sigma_1 \circ h_1 \circ \sigma_2^{-1}. \end{aligned}$$

We denote $(f_1, g_1, h_1) \sim (f_2, g_2, h_2)$.

The following theorem tells us that one can always apply a reduction step to assume that $\deg f = \deg g$ and $\deg h$ are co-prime.

Theorem 4.5. *Let $[(f, g, h)] \in \mathcal{S}$. If $\gcd(\deg f, \deg h) = d > 1$, then there exist polynomials $g_1, h_1, f_1, \hat{h}_1, \alpha_1$ and β_1 such that*

$$f \circ h_1 = h_1 \circ g_1, \quad f_1 \circ \hat{h}_1 = \hat{h}_1 \circ g, \quad h = h_1 \circ \beta_1 = \alpha_1 \circ \hat{h}_1,$$

$$\deg f = \deg g_1 = \deg f_1, \quad \deg \alpha_1 = \deg \beta_1 = d, \quad \deg h_1 = \deg \hat{h}_1 = \deg h/d.$$

In particular, if $d < \deg h$, then $[(f, g_1, h_1)], [(f_1, g, \hat{h}_1)] \in \mathcal{S}$.

The next theorem gives a complete classification for the case $\gcd(\deg f, \deg h) = 1$.

Theorem 4.6. *Assume that $[(f, g, h)] \in \mathcal{S}$ satisfies $\gcd(\deg f, \deg h) = 1$. Then there exists a representative (f_0, g_0, h_0) of $[(f, g, h)]$ such that one of the following is true:*

- $f(z) = z^r (P(z))^b$, $g(z) = z^r P(z^b)$ and $h(z) = z^b$, where $r = a \bmod b$, and P is a complex polynomial,
- $f = g = T_a$, $h = T_b$ are Chebyshev polynomials (of degree a and b respectively),

where $a = \deg f (= \deg g)$ and $b = \deg h$.

Proof of Theorem 1.2. If the two semi-conjugacies appearing in Lemma 4.4 are affine conjugacies (i.e. if p_1 and p_2 have degree 1), then $f_{\tilde{c}}^{\circ 2k}$ and $f_{\tilde{c}^*}^{\circ 2k}$ are affinely conjugate, and a straightforward computation shows that $\tilde{c}^* = \omega^j \tilde{c}$ where $\omega = \exp(\frac{2\pi i}{d+1})$, and $j \in \mathbb{N}$. Setting $\tilde{c} = re^{2\pi i \vartheta}$, we see that $\vartheta \in \mathbb{Z}/2(d+1)$. We need to consider two cases now. When d is even, the condition $\vartheta \in \mathbb{Z}/2(d+1)$ implies that there is some integer k such that either $\vartheta + \frac{k}{d+1} = 0$ or $\vartheta + \frac{k}{d+1} = \frac{1}{2}$. Therefore when d is even, $f_{\tilde{c}}$ is affinely conjugate to some f_c with $c \in \mathbb{R}$. Now let us consider the case when d is odd. In this case, the condition $\vartheta \in \mathbb{Z}/2(d+1)$ does not necessarily imply that $f_{\tilde{c}}$ is affinely conjugate to some f_c with $c \in \mathbb{R}$. However for odd degree multicorns, the situation is rather restricted. If $f_{\tilde{c}}$ has a parabolic cycle for some $\tilde{c} = re^{2\pi i \vartheta}$ with $\vartheta \in \mathbb{Z}/(d+1)$ (i.e. $f_{\tilde{c}}$ is affinely conjugate to a real antiholomorphic polynomial), then \tilde{c} lies on a period 1 parabolic arc. Recall that by [MNS15, Lemma 5.3], each parabolic arc of period 1 is a co-root arc, and hence \tilde{c} cannot be the landing point of a path $p : [0, \delta] \rightarrow \mathbb{C}$ with $p(0) = \tilde{c}$, and $p((0, \delta]) \subset \mathcal{M}_d^* \setminus \overline{H_0}$ (where H_0 is the hyperbolic component of period 1). On the other hand, if $f_{\tilde{c}}$ has a parabolic cycle for some $\tilde{c} = re^{2\pi i \vartheta}$ with $\vartheta \in \mathbb{Z}/2(d+1) \setminus \mathbb{Z}/(d+1)$, then \tilde{c} is either a parabolic cusp of period 1, or a co-root point of a hyperbolic component of period 2. In particular, such a \tilde{c} cannot lie on a root arc of an odd period hyperbolic component of \mathcal{M}_d^* . This completes the proof of the theorem in the case when both p_1 and p_2 are of degree 1.

Therefore, we only need to deal with the situation $\deg p_1 = \deg p_2 = b > 1$. We will first prove by contradiction that $\gcd(\deg f_{\tilde{c}}^{\circ 2k}, \deg p_1) > 1$. To do this, let $\gcd(\deg f_{\tilde{c}}^{\circ 2k}, \deg p_1) = 1$, i.e. $\gcd(d^{2k}, b) = 1$, i.e. $\gcd(d, b) = 1$. Now we can apply Theorem 4.6 to our situation; but since $f_{\tilde{c}}^{\circ 2k}$ is parabolic, it is neither a power map, nor a Chebyshev polynomial. Hence, there exists some non-constant polynomial P such that $f_{\tilde{c}}^{\circ 2k}$ is affinely conjugate to the polynomial $g(z) := z^r (P(z))^b$. If $r \geq 2$, then $g(z)$ has a super-attracting

fixed point at 0. But $f_{\tilde{c}}^{\circ 2k}$, which is affinely conjugate to $g(z)$, has no super-attracting fixed point. Hence, $r = 0$ or 1. By degree consideration, we have $d^{2k} = r + bk$, where $\deg P = k$. The assumption $\gcd(d, b) = 1$ implies that $r = 1$, i.e. $g(z) = z(P(z))^b$. Now the fixed point 0 for g satisfies $g^{-1}(0) = \{0\} \cup P^{-1}(0)$, and any point in $P^{-1}(0)$ has a local mapping degree b under g . The same must hold for the affinely conjugate polynomial $f_{\tilde{c}}^{\circ 2k}$: there exists a fixed point (say x) for $f_{\tilde{c}}^{\circ 2k}$ such that any point in $(f_{\tilde{c}}^{\circ 2k})^{-1}(x)$ has mapping degree b except for x ; in particular, all points in $(f_{\tilde{c}}^{\circ 2k})^{-1}(x) \setminus \{x\}$ are critical points for $f_{\tilde{c}}^{\circ 2k}$ (since $b > 1$). However, the local degree of any critical point for $f_{\tilde{c}}^{\circ 2k}$ is equal to d^r for some $r \geq 1$ (every critical point of $f_{\tilde{c}}^{\circ 2}$ has mapping degree d); so $b = d^r$, and this contradicts the assumption that $\gcd(d, b) = 1$ (alternatively, we could use the fact that $f_{\tilde{c}}^{\circ 2k}$ has no finite critical orbit).

Applying Engstrom's theorem [Eng41] (compare [Ino11, Theorem 11, Corollary 12, Lemma 13]), we obtain the existence of polynomials (of degree at least two) g, h, g_1 and h_1 such that up to affine conjugacy

$$f_{\tilde{c}}^{\circ 2k} = h \circ g, \quad f_{\tilde{c}^*}^{\circ 2k} = h_1 \circ g_1, \quad p = g \circ h = g_1 \circ h_1, \quad \text{and} \quad \deg(g) = \deg(g_1).$$

The equation $f_{\tilde{c}}^{\circ 2k} = h \circ g$ implies that $f_{\tilde{c}^*}^{\circ 2k} = (\iota \circ h \circ \iota) \circ (\iota \circ g \circ \iota)$. Note that the only possible non-uniqueness in the decomposition of $f_{\tilde{c}^*}^{\circ 2k}$ into prime factors (under composition) occurs due to the relation

$$z^d + \tilde{c} = (z^{d_1} + \tilde{c}) \circ (z^{d_2}) = (z^{d_2} + \tilde{c}) \circ (z^{d_1}) \quad \text{with } d = d_1 d_2, \quad d_1, d_2 \geq 2.$$

However, we claim that $h_1 = \iota \circ h \circ \iota$, and $g_1 = \iota \circ g \circ \iota$. Indeed, if h_1 and $\iota \circ h \circ \iota$ (and hence g_1 and $\iota \circ g \circ \iota$) have different decompositions, then using the type of non-uniqueness, the relation $p = g \circ h = g_1 \circ h_1$, and the fact that $\deg(\iota \circ g \circ \iota) = \deg(g) = \deg(g_1)$, one obtains two different sets of multiplicities of the critical points for the same polynomial p . This contradiction proves the claim.

Therefore, $p = g \circ h = (\iota \circ g \circ \iota) \circ (\iota \circ h \circ \iota)$ (up to affine conjugacy). Hence g and h are real polynomials, implying that $g_1 = g$, and $h_1 = h$. It now follows that $f_{\tilde{c}}^{\circ 2k}$ and $f_{\tilde{c}^*}^{\circ 2k}$ are affinely conjugate. We can now argue as in the first paragraph of the proof to conclude that d is even, and $\tilde{c} \in \mathbb{R} \cup \omega\mathbb{R} \cup \omega^2\mathbb{R} \cup \dots \cup \omega^d\mathbb{R}$, where $\omega = \exp(\frac{2\pi i}{d+1})$. \square

Corollary 4.7. *For odd d , all umbilical cords wiggle.*

Corollary 4.8. *Let c be an odd period non-cusp parabolic parameter of \mathcal{M}_d^* with critical Ecalle height 0. If the characteristic parabolic germ of f_c is conformally conjugate to a real parabolic germ, then $c^* = \omega^j c$ for some $j \in \{0, 1, \dots, d\}$, where $\omega = \exp(\frac{2\pi i}{d+1})$. In other words, f_c commutes with the global antiholomorphic involution $\zeta \mapsto \omega^{-j} \zeta^*$.*

Remark. We should point out that Theorem 1.2 shows that if a hyperbolic component H of odd period (different from 1) of \mathcal{M}_d^* does not intersect the real line or its ω -rotates, then H cannot be connected to the principal

hyperbolic component (of period 1) by a path inside of \mathcal{M}_d^* . This statement is a sharper version of a result of Hubbard and Schleicher on the non path-connectedness of the multicorns [HS14, Theorem 6.2].

5. RENORMALIZATION, AND MULTICORN-LIKE SETS

In this section, we will give a brief overview of the combinatorics and topology of straightening maps in consistence with [IK12]. After preparing the necessary background on renormalization and straightening, we will introduce the concepts of ‘multicorn-like sets’, and the ‘straightening map’ from ‘multicorn-like sets’ to the actual multicorn (compare [Ino14]). Finally, we will state the principal results of [IK12], applied to our setting.

Definition (Polynomial-like and Anti-polynomial-like Maps). We call a map $g : U' \rightarrow U$ *polynomial-like* (respectively *anti-polynomial-like*) if

- U', U are topological disks in \mathbb{C} , and U' is relatively compact in U .
- $g : U' \rightarrow U$ is holomorphic (respectively antiholomorphic), and proper.

The *filled-in Julia* set $K(g)$ and the *Julia set* $J(g)$ are defined as follows:

$$K(g) = \{z \in U' : g^{on}(z) \in U', \forall n \in \mathbb{N}\}, J(g) = \partial K(g).$$

In particular, we say that polynomial-like (or anti-polynomial-like) mapping is *unicritical-like* if it has a unique critical point of possibly higher multiplicity. The importance of (anti-)polynomial-like maps stems from the fact that they behave, in a certain sense, like (anti-)polynomials. This is justified by the following Straightening Theorem [DH85, Theorem 1] which is proved in the same way as in the holomorphic case: every anti-polynomial-like map of degree d is hybrid equivalent to an anti-polynomial of equal degree.

Definition (Hybrid Equivalence). Two polynomial-like (or anti-polynomial-like) mappings $f : U' \rightarrow U$ and $g : V' \rightarrow V$ are *hybrid equivalent* if there exists a quasi-conformal homeomorphism $\varphi : U'' \rightarrow V''$ between neighborhoods U'' and V'' of $K(f)$ and $K(g)$ respectively, such that $\varphi \circ f = g \circ \varphi$ whenever both sides are defined, and $\bar{\partial}\varphi = 0$ almost everywhere in $K(f)$.

Theorem 5.1 (Straightening Theorem). *Any polynomial-like (respectively anti-polynomial-like) mapping $g : U' \rightarrow U$ is hybrid equivalent to a holomorphic (respectively antiholomorphic) polynomial P of the same degree. Moreover, if $K(g)$ is connected, then P is unique up to affine conjugacy.*

Remark. We define the degree of g as the number of pre-images of any point, so it is always positive. Hence, for an antiholomorphic map g , d is the degree (in the classical sense) of the proper holomorphic map $g^* : U \rightarrow V^*$ which is the complex conjugate of g .

Definition (Renormalization, and Straightening). We say f_c is *renormalizable* if there exist U'_c, U_c (containing the critical point 0), and $k > 1$ such that $f_c^{ok} : U'_c \rightarrow U_c$ is unicritical-like, and has a connected filled-in Julia set.

Such a mapping $f_c^{\circ k} : U'_c \rightarrow U_c$ is called a *renormalization* of f_c , and k is called its *period*.

By the straightening theorem, there exists a unique monic centered holomorphic or antiholomorphic unicritical polynomial P hybrid equivalent to $f_c^{\circ k} : U'_c \rightarrow U_c$, up to affine conjugacy. We call P the *straightening* of the renormalization.

Take $c_0 \in \mathcal{M}_d^*$ such that 0 is a periodic point of period $k > 1$ of f_{c_0} ; i.e. c_0 is a center (of a hyperbolic component of $\text{int}(\mathcal{M}_d^*)$) of period k . Let $\lambda_0 = \lambda(f_{c_0})$ be the rational lamination of f_{c_0} . Define the combinatorial renormalization locus $\mathcal{C}(c_0)$ as follows:

$$\mathcal{C}(c_0) = \{c \in \mathcal{M}_d^* : \lambda(f_c) \supset \lambda_0\}.$$

Since a rational lamination is an equivalence relation on \mathbb{Q}/\mathbb{Z} , it is a subset of $\mathbb{Q}/\mathbb{Z} \times \mathbb{Q}/\mathbb{Z}$, and hence, subset inclusion makes sense. By definition, for $c \in \mathcal{C}(c_0)$, the external rays of λ_0 -equivalent angles for f_c land at the same point. Hence those rays divide K_c into ‘*fibers*’. Let K be the fiber containing the critical point 0. Then $f_c^{\circ k}(K) = K$. We say f_c is *c_0 -renormalizable* if there exists a (holomorphic or antiholomorphic) unicritical-like restriction $f_c^{\circ k} : U'_c \rightarrow U_c$ such that the filled-in Julia set is equal to K . Let the renormalization locus $\mathcal{R}(c_0)$ with combinatorics λ_0 be:

$$\mathcal{R}(c_0) = \{c \in \mathcal{C}(c_0) : f_c \text{ is } c_0\text{-renormalizable}\}.$$

We call such a renormalization a *c_0 -renormalization* (see the definition of *λ_0 -renormalization* in [IK12] for a more general definition). We call k the *renormalization period*.

For the rest of this section, we fix such a c_0 , and its rational lamination $\lambda_0 = \lambda(f_{c_0})$. For $c \in \mathcal{R}(c_0)$, let P be the straightening of a c_0 -renormalization of f_c . By the straightening theorem, P is well-defined. When the renormalization period k is even, then the c_0 -renormalization is holomorphic. Hence, $P = f_{c'}$ for some $c' \in \mathcal{M}_d$. When k is odd, the c_0 -renormalization is antiholomorphic, so $P = f_{c'}$ for some $c' \in \mathcal{M}_d^*$. In either case, we denote c' by $\chi_{c_0}(c)$ (here, we have tacitly fixed an external marking for our (anti-)polynomial-like maps so that the map χ_{c_0} is well-defined).

To relate our definition of χ_{c_0} with the general notion of straightening for holomorphic polynomials (as developed in [IK12]), we need to work with $P_{\bar{c},c} = f_c^{\circ 2}$. This allows us to embed $\{f_c\}_{c \in \mathbb{C}}$ in the family $\text{Poly}(d^2)$ of monic centered polynomials of degree d^2 . We will denote the real 2-dimensional plane in which $\text{Poly}(d^2)$ intersects the family $\{P_{\bar{c},c}\}_{c \in \mathbb{C}}$ by L . Since $P_{\bar{c}_0,c_0}$ is a post-critically finite hyperbolic polynomial (of degree d^2) with rational lamination λ_0 , we are now in the setting of renormalization and straightening maps defined over reduced mapping schemas. We refer the readers to [IK12, §1] for these general notions.

The combinatorial renormalization locus

$$\mathcal{C}(\lambda_0) := \{g \in \text{Poly}(d^2) : \lambda(g) \supset \lambda_0\},$$

and the renormalization locus

$$\mathcal{R}(\lambda_0) = \{g \in \mathcal{C}(\lambda_0) : g \text{ is } \lambda_0\text{-renormalizable}\}$$

satisfy $\mathcal{C}(c_0) = \mathcal{C}(\lambda_0) \cap L$, and $\mathcal{R}(c_0) = \mathcal{R}(\lambda_0) \cap L$. There is a straightening map $\chi_{\lambda_0} : \mathcal{R}(\lambda_0) \rightarrow \mathcal{C}(T(\lambda_0))$, where $\mathcal{C}(T(\lambda_0))$ is the fiber-wise connectedness locus of the family of monic centered polynomial maps over the reduced mapping scheme $T(\lambda_0)$ of λ_0 . Following [Ino14], we will now describe the set $\mathcal{C}(T(\lambda_0))$.

When k is even, $P_{\overline{c}_0, c_0}$ has two disjoint periodic cycles each containing a single critical point of multiplicity d (disjoint mapping scheme). This gives rise to two independent holomorphic unicritical-like maps (of degree d), and hence $\mathcal{C}(T(\lambda_0)) = \mathcal{M}_d \times \mathcal{M}_d$, which is the fiber-wise connectedness locus of the family:

$$\begin{aligned} \{P : \{0, 1\} \times \mathbb{C} \curvearrowright; P(k, z) = (k, p_{a_k}(z)), p_{a_k}(z) = z^d + a_k, a_k \in \mathbb{C}\} \\ = \{(p_{a_0}, p_{a_1}) : a_0, a_1 \in \mathbb{C}\} \cong \mathbb{C}^2. \end{aligned}$$

Now let $c \in \mathcal{R}(c_0)$. Every c_0 -renormalization $f_c^{\circ k} : U'_c \rightarrow U_c$ splits into two (holomorphic) unicritical-like maps (of degree d) $P_{\overline{c}, c}^{\circ k/2} : U'_c \rightarrow U_c$ and $P_{\overline{c}, c}^{\circ k/2} : f_c(U'_c) \rightarrow f_c(U_c)$ (after shrinking U'_c and U_c if necessary). Moreover, the former (holomorphic) unicritical-like restriction is antiholomorphically conjugate to the latter one by $f_c^{\circ(k-1)}$ near the filled-in Julia sets (note that since k is even, $f_c^{\circ(k-1)}$ is antiholomorphic). Therefore, as λ_0 -renormalization for $P_{\overline{c}, c}$, we have two (holomorphic) unicritical-like maps of degree d which are antiholomorphically equivalent. After fixing an external marking for our polynomial-like maps, we conclude that the straightening of $P_{\overline{c}, c}^{\circ k/2} : U'_c \rightarrow U_c$ and $P_{\overline{c}, c}^{\circ k/2} : f_c(U'_c) \rightarrow f_c(U_c)$ are of the form $p_{c'}$ and $p_{\overline{c'}}$ (recall that $p_c(z) = z^d + c$). Therefore, modulo a fixed choice of external marking, for any $c \in \mathcal{R}(c_0)$ we have that $\chi_{\lambda_0}(P_{\overline{c}, c}) = (p_{c'}, p_{\overline{c'}})$, for a unique $c' \in \mathcal{M}_d$ (by the condition of having a connected filled-in Julia set). On the other hand, the c_0 -renormalization $f_c^{\circ k} : U'_c \rightarrow U_c$ is holomorphic and unicritical-like of degree d . By the definition of straightening, $\chi_{c_0}(f_c) = p_{c'}$.

Now let k be odd. Then both the periodic critical points (of multiplicity d) of $P_{\overline{c}_0, c_0}$ lie on the same cycle (bitransitive mapping scheme). For any $c \in \mathcal{R}(c_0)$, the quartic-like map $P_{\overline{c}, c}^{\circ k} : U'_c \rightarrow P_{\overline{c}, c}^{\circ k}(U'_c) = f_c^{\circ k}(U_c)$ can be written as the composition of the two unicritical holomorphic maps

$$Q_1 : P_{\overline{c}, c}^{\circ \frac{k-1}{2}} : U'_c \rightarrow f_c^{\circ(k-1)}(U'_c), \quad Q_2 : P_{\overline{c}, c}^{\circ \frac{k+1}{2}} : f_c^{\circ(k-1)}(U'_c) \rightarrow f_c^{\circ k}(U_c),$$

each of degree d . Hence it follows that the straightening is a composition of two degree d unicritical polynomials. Therefore, $\mathcal{C}(T(\lambda_0))$ is the fiber-wise connectedness locus of the family:

$$\begin{aligned}
\text{Poly}(d \times d) &= \{P : \{0, 1\} \times \mathbb{C} \circlearrowleft; P(k, z) = (1 - k, p_{a_k}(z)), \\
&\quad p_{a_k}(z) = z^d + a_k, a_k \in \mathbb{C}\} \\
&= \{(p_{a_0}, p_{a_1}) : a_0, a_1 \in \mathbb{C}\} \cong \mathbb{C}^2.
\end{aligned}$$

Identifying any $P \in \text{Poly}(d \times d)$ with the composition $p_{a_1} \circ p_{a_0}$, we can view $\mathcal{C}(T(\lambda_0))$ as the connectedness locus of the family $\{(z^d + a)^d + b\}_{a, b \in \mathbb{C}}$.

Now let $c \in \mathcal{R}(c_0)$, and $\chi_{\lambda_0}(P_{\bar{c}, c}) = (p_{a_0}, p_{a_1})$. Note that

$$\begin{aligned}
Q_2 \circ Q_1 &= P_{\bar{c}, c}^{\circ k} : U'_c \rightarrow f_c^{\circ k}(U_c) \text{ and} \\
Q_1 \circ Q_2 &= P_{\bar{c}, c}^{\circ k} : f_c^{\circ(k-1)}(U'_c) \rightarrow f_c^{\circ(2k-1)}(U_c)
\end{aligned}$$

are antiholomorphically conjugate by f_c near their filled-in Julia sets. Therefore, the straightenings of $Q_2 \circ Q_1$ and $Q_1 \circ Q_2$ are conjugate by an affine antiholomorphic map. Hence, they satisfy $(p_{\bar{a}_0}, p_{\bar{a}_1}) = (p_{a_1}, p_{a_0})$. Therefore, $\chi_{\lambda_0}(P_{\bar{c}, c}) = (p_{a_0}, p_{\bar{a}_0})$. Using the identification of $\text{Poly}(d \times d)$ with maps of the form $(z^d + a)^d + b$, we obtain that $\chi_{\lambda_0}(P_{\bar{c}, c}) = p_{a_0} \circ p_{\bar{a}_0} = f_{a_0}^{\circ 2}$, for a unique $a_0 \in \mathcal{M}_d^*$, once we have fixed an external marking for our (anti-)polynomial-like maps (by the condition of having a connected filled-in Julia set). On the other hand, the c_0 -renormalization $f_c^{\circ k} : U'_c \rightarrow U_c$ is antiholomorphic and uncritical-like of degree d . By the definition of straightening, $\chi_{c_0}(f_c) = f_{a_0}$.

The above discussion (along with our chosen identifications) shows that the maps χ_{c_0} and χ_{λ_0} are essentially the same on $\mathcal{R}(c_0)$.

Define

$$\mathcal{M}(c_0) = \begin{cases} \mathcal{M}_d & \text{if } k \text{ is even,} \\ \mathcal{M}_d^* & \text{if } k \text{ is odd.} \end{cases}$$

Definition (Straightening Map). We call the map $\chi_{c_0} : \mathcal{R}(c_0) \rightarrow \mathcal{M}(c_0)$ as above the *straightening map* for c_0 .

We call $\mathcal{C}(c_0)$ a *baby multibrot-like set* when the renormalization period is even. Otherwise, we call it a *baby multicorn-like set*.

By the rotational symmetry of the multicorns, if the period k is odd, then $\omega\chi_{c_0}, \omega^2\chi_{c_0}, \dots, \omega^d\chi_{c_0}$ are also straightening maps (with different internal/external markings), where $\omega = \exp(\frac{2\pi i}{d+1})$. In the sequel, we will always choose, and fix one of them.

Definition. We call a center $c_0 \in \mathcal{M}_d^*$ primitive if the closures of Fatou components of f_{c_0} are mutually disjoint.

With these preparations, we are now ready to state the main results from [IK12] applied to our setting. Strictly speaking, these theorems hold for the map χ_{λ_0} , but we can apply them to the map χ_{c_0} since these two maps, suitably interpreted, agree on $\mathcal{R}(c_0)$.

Theorem 5.2 (Injectivity). *The straightening map $\chi_{c_0} : \mathcal{R}(c_0) \rightarrow \mathcal{M}(c_0)$ is injective.*

Theorem 5.3 (Onto Hyperbolicity). *The image $\chi_{c_0}(\mathcal{R}(c_0))$ of the straightening map contains all the hyperbolic components of $\mathcal{M}(c_0)$.*

Theorem 5.4 (Compactness). *If c_0 is primitive, then $\mathcal{C}(c_0) = \mathcal{R}(c_0)$, and it is compact.*

These general theorems also imply that in the even period case, the straightening map from a baby multibrot-like set to the original multibrot set is a homeomorphism (at least in good cases). See [Ino14, Appendix A] for a proof.

Theorem 5.5. *If c_0 is primitive, and the renormalization period is even, then the corresponding straightening map $\chi_{c_0} : \mathcal{R}(c_0) \rightarrow \mathcal{M}_d$ is a homeomorphism.*

6. A CONTINUITY PROPERTY OF STRAIGHTENING MAPS

According to [IK12, Theorem C], the straightening map χ , restricted to any hyperbolic component of \mathcal{M}_d^* , is a real-analytic homeomorphism. χ admits a homeomorphic extension to the boundaries of even period hyperbolic components under the conditions of Theorem 5.5. In this section, we will study the corresponding property of χ when the renormalization period is odd. In fact, we will show that in this case, χ always extends as a homeomorphism to the boundaries of odd period hyperbolic components.

We would like to thank John Milnor for bringing this question (a precursor to which can be found in one of the author's thesis [Muk15a, Appendix B]) to our attention. In the special case $d = 2$, this has been independently answered in [BBM, §3], and our treatment borrows heavily from their discussion. Since the terminology, and the parameter spaces under consideration in [BBM] differ from ours, it is worthwhile to record the results here.

Let H be a hyperbolic component of odd period k of \mathcal{M}_d^* , and let c_0 be the center of H . Let $c \in H \setminus \{c_0\}$, and z_0 be the attracting periodic point of f_c contained in the critical value Fatou component U_c .

So, $f_c^{\circ k}(z_0) = z_0$. Let, $\frac{\partial f_c^{\circ k}}{\partial \bar{z}}(z_0) = \lambda_c$.

The multiplier $\frac{\partial f_c^{\circ 2k}}{\partial \bar{z}}(z_0)$ is:

$$\begin{aligned} \frac{\partial f_c^{\circ 2k}}{\partial \bar{z}}(z_0) &= \frac{\partial f_c^{\circ k}}{\partial \bar{z}}(z_0) \overline{\frac{\partial f_c^{\circ k}}{\partial \bar{z}}(z_0)} \\ &= \left| \frac{\partial f_c^{\circ k}}{\partial \bar{z}}(z_0) \right|^2 \\ &= |\lambda_c|^2. \end{aligned}$$

Following [BBM, Definition 3.4] or [Muk15a, Appendix B], we will associate a conformal invariant to f_c . In fact, the notion is similar to that of Ecalle height of a parabolic parameter. In our situation, there are two distinct critical orbits (for the second iterate $f_c^{\circ 2}$) converging to an attracting cycle. One can choose two representatives of these two critical orbits

in a fundamental domain (in the critical value Fatou component), and consider their ratio under a holomorphic Koenigs' coordinate. More precisely, if $\kappa_c : U_c \rightarrow \mathbb{C}$ is a Koenigs linearizing coordinate for the unique attracting periodic point of f_c in U_c with $\kappa_c(f_c^{\circ 2k}(z)) = |\lambda_c|^2 \kappa_c(z)$, then we define an invariant

$$\rho_H(c) := \frac{\kappa_c(f_c^{\circ k}(c))}{\kappa_c(c)}.$$

At the center c_0 , we define $\rho_H(c_0) = 0$.

This ratio is well-defined as the choice of Koenig's coordinate doesn't affect it, and hence is a conformal invariant of f_c . Moreover, $|\rho_H(c)| = |\lambda_c|$, so $|\rho_H(c)| \rightarrow +1$ as $c \rightarrow \partial H$. We will call it the *Koenigs ratio*. It is proved in [BBM, Lemma 3.5] that $\rho_H : H \rightarrow \mathbb{D}$ is a real-analytic $(d+1)$ -fold branched covering branched only at origin (they prove it for the tricorn, the degree d case follows directly). Alternatively, it is easy to see that $\rho_H(c)$ agrees with the 'critical value map' for odd period hyperbolic components introduced in [NS03, §5].

Definition (Internal Rays of Odd Period Components). An *internal ray* of an odd period hyperbolic component H of \mathcal{M}_d^* is an arc $\gamma \subset H$ starting at the center such that there is an angle ϑ with $\rho_H(\gamma) = \{re^{2\pi i\vartheta} : r \in [0, 1)\}$.

Remark. Since ρ_H is a $(d+1)$ -to-one map, an internal ray of H with a given angle is not uniquely defined. In fact, a hyperbolic component has $(d+1)$ internal rays with any given angle ϑ .

Let \tilde{c} be a non-cusp parabolic parameter on the boundary of H . To understand the landing behavior of the internal rays, we will now relate the Koenigs ratio of f_c to the critical Ecalle height of $f_{\tilde{c}}$ as c approaches \tilde{c} . We have the following lemmas.

Lemma 6.1 (Relation between Koenigs Ratio, and Ecalle Height). *As c in H approaches a non-cusp parabolic parameter with critical Ecalle height h on the boundary of H , the quantity $\frac{1-\rho_H(c)}{1-|\rho_H(c)|^2}$ converges to $\frac{1}{2} - 2ih$.*

Proof. See [BBM, Lemma 3.9] for a proof.

Alternatively, set $S_c(w) = \frac{|\lambda_c|^2(w-1)}{(|\lambda_c|^2-1)w}$. A direct computation shows that $S_c \circ \kappa_c(f_c^{\circ k}(c)) - S_c \circ \kappa_c(c) = \frac{1-\rho_H(c)}{1-|\rho_H(c)|^2}$, and $\psi_c^{\text{att}}(f_c^{\circ k}(c)) - \psi_c^{\text{att}}(c) = \frac{1}{2} - 2ih$. Now using [Kaw07, Theorem 1.2], we obtain the limiting relation between two conformal invariants as c approaches the parabolic parameter \tilde{c} . \square

The landing properties of the internal rays follow directly from the above lemma.

Lemma 6.2 (Internal Rays Land). *The $d+1$ internal rays at angle 0 land at the $d+1$ critical Ecalle height 0 parameters on ∂H (one ray on each parabolic arc). All other internal rays land at the cusp points on ∂H .*

Proof. Let γ be an internal ray at angle ϑ , and \tilde{c} be an accumulation point of γ on ∂H . Further assume that the critical Ecalle height of $f_{\tilde{c}}$ be h . As c approaches \tilde{c} (along γ), $|\rho_H(c)|$ goes to $+1$, and $\frac{1-|\rho_H(c)|e^{2\pi i\vartheta}}{1-|\rho_H(c)|^2}$ converges to $\frac{1}{2} - 2ih$ (by Lemma 6.1). It follows that $\vartheta = 0$. But for $\vartheta = 0$, we have $\frac{1-|\rho_H(c)|}{1-|\rho_H(c)|^2} = \frac{1}{1+|\rho_H(c)|} \rightarrow \frac{1}{2}$ as $|\rho_H(c)|$ goes to $+1$, i.e. as c goes to ∂H . This shows that the only accumulation point of the internal rays at angle 0 are the critical Ecalle height 0 parameters; i.e. these rays land there (note that there are only finitely many critical Ecalle height 0 parameters on ∂H). On the other hand, the above argument shows that no internal ray at an angle $\vartheta \neq 0$ can accumulate at non-cusp parameters. Since \overline{H} is compact, and ∂H consists of $d+1$ parabolic arcs (of non-cusp parameters), and $d+1$ cusp points, it follows that every internal ray at angle ϑ different from 0 lands at a cusp on ∂H . \square

Finally, note that the limiting relation between Koenigs ratio and Ecalle height obtained in Lemma 6.1 holds uniformly for any hyperbolic component of odd period of \mathcal{M}_d^* . Since the straightening map χ preserves these conformal invariants, we have the following theorem.

Theorem 6.3 (Homeomorphism between Closures of Odd Period Hyperbolic Components). *Let c_0 be the center of a hyperbolic component of odd period of \mathcal{M}_d^* (respectively, of a tricorn-like set in the real cubic locus). Then $\chi : \mathcal{R}(c_0) \rightarrow \mathcal{M}_d^*$ (respectively, $\chi : \mathcal{R}(c_0) \rightarrow \mathcal{M}_2^*$) restricted to the closure $\overline{H'}$ of any odd period hyperbolic component $H' \subset \mathcal{R}(c_0)$ is a homeomorphism.*

7. DISCONTINUITY OF STRAIGHTENING MAPS

In this section, we will continue our study of straightening maps as developed in the previous section, and will prove our main theorem on the discontinuity of straightening maps in the odd period case.

It follows from an argument similar to Lemma 3.1 that the centers of odd period hyperbolic components of \mathcal{M}_d^* are primitive. Using Theorem 5.4, we conclude that:

Corollary 7.1. *If the renormalization period is odd, then $\mathcal{C}(c_0) = \mathcal{R}(c_0)$, and it is compact.*

Corollary 7.2. *If the renormalization period is odd, then the image $\chi_{c_0}(\mathcal{R}(c_0))$ of the straightening map contains the real part $\mathbb{R} \cap \mathcal{M}_d^*$.*

Proof. The proof is similar to [Ino14, Corollary 5.3], where this has been proved in the quadratic case. For the degree d case, one needs to use density of hyperbolicity of real polynomials proved in [KSS07] [AKLS09]. \square

Before giving the proof of the main theorem of this paper, we need to show the existence of hyperbolic components in the real part of \mathcal{M}_d^* .

Lemma 7.3. *If d is even, then at least one hyperbolic component of period 3 of \mathcal{M}_d^* intersects $\mathbb{R} \cap \mathcal{M}_d^*$.*

Proof. Let, S_d be the set of all hyperbolic components of period 3 of \mathcal{M}_d^* . By [MNS15, Theorem 1.3, Lemma 7.1], $|S_d| = d^2 - 1$. Note that \mathcal{M}_d^* has a complex conjugation symmetry; i.e. complex conjugation induces an involutive bijection on the set S_d . When d is even, $|S_d|$ is an odd integer, and this implies that there must be some element H_d^* in S_d fixed by complex conjugation. Clearly, H_d^* intersects the real line. Moreover, the center of H_d^* , and the critical Ecalle height 0 parameter on the root arc of ∂H_d^* are real, and a piece of $\mathbb{R} \cap \mathcal{M}_d^*$ converges to this critical Ecalle height 0 parameter from the exterior of $\overline{H_d^*}$. \square

Proof of Theorem 1.1. Let, d be even, and c_0 be the center of a hyperbolic component of odd period k ($k \neq 1$) of \mathcal{M}_d^* . We will assume that the map $\chi_{c_0} : \mathcal{R}(c_0) \rightarrow \mathcal{M}_d^*$ is continuous, and will arrive at a contradiction.

By Corollary 7.1 and Theorem 5.2, $\mathcal{R}(c_0)$ is compact, and the map χ_{c_0} is injective. Since an injective continuous map from a compact topological space onto a Hausdorff topological space is a homeomorphism, it follows that χ_{c_0} is a homeomorphism from $\mathcal{R}(c_0)$ onto its range (we do not claim that $\chi_{c_0}(\mathcal{R}(c_0)) = \mathcal{M}_d^*$). It follows from Corollary 7.2 (and by symmetry), that $(\mathbb{R} \cup \omega\mathbb{R} \cdots \cup \omega^d\mathbb{R}) \cap \mathcal{M}_d^* \subset \chi_{c_0}(\mathcal{R}(c_0))$. Moreover, by Theorem 5.3, $H_d^* \cup \omega H_d^* \cup \cdots \cup \omega^d H_d^* \subset \chi_{c_0}(\mathcal{R}(c_0))$.

Since c_0 is not of period 1, there exists $i \in \{0, 1, \dots, d\}$ such that $H' := \chi_{c_0}^{-1}(\omega^i H_d^*)$ does not intersect the real line or its ω -rotates (but is contained in $\mathcal{R}(c_0)$). Recall that there exists a piece γ of $\omega^i \mathbb{R} \cap \mathcal{M}_d^*$ that lies outside of $\overline{\omega^i H_d^*}$, and lands at the critical Ecalle height 0 parameter on the root parabolic arc of $\partial(\omega^i H_d^*)$. By our assumption, χ_{c_0} is a homeomorphism; and hence the curve $\chi_{c_0}^{-1}(\gamma)$ lies in the exterior of $\overline{H'}$, and lands at the critical Ecalle height 0 parameter on the root arc of $\partial H'$ (critical Ecalle heights are preserved by hybrid equivalences). Since H' does not intersect the real line or its ω -rotates, this contradicts Theorem 1.2.

The hyperbolic component of period 3 (intersecting the real line) does not play any special role in the above proof; in fact, there are infinitely many odd period hyperbolic components of \mathcal{M}_d^* that intersect the real line. Our argument applies verbatim to any of these hyperbolic components, which proves that straightening maps are indeed discontinuous at infinitely many parameters. \square

8. TRICORNS IN REAL CUBICS

In this section, we will discuss some topological properties of tricorn-like sets, and umbilical cords in the family of real cubic polynomials. We will work with the family:

$$\mathcal{G} = \{g_{a,b}(z) = -z^3 - 3a^2z + b, a \geq 0, b \in \mathbb{R}\}.$$

Milnor [Mil92] numerically found that the connectedness locus of this family contains tricorn-like sets. We will rigorously define tricorn-like sets

in this family (via straightening of suitable anti-polynomial-like maps), and show that the corresponding straightening maps from these tricorn-like sets to the original tricorn are discontinuous.

$g_{a,b}$ commutes with complex conjugation ι ; i.e. $g_{a,b}$ has a reflection symmetry with respect to the real line. Observe that $g_{a,b}$ is conjugate to the monic centered polynomial $h_{a,b}(z) = z^3 - 3a^2z + bi$ by the affine map $z \mapsto iz$, and $h_{a,b}$ has a reflection symmetry with respect to the imaginary axis (i.e. $h_{a,b}(-\iota(z)) = -\iota(h_{a,b}(z))$, where $\iota(z) = \bar{z}$). We will, however, work with the real form² $g_{a,b}$, and will normalize the Böttcher coordinate $\varphi_{a,b}$ of $g_{a,b}$ (at ∞) such that $\varphi_{a,b}(z)/z \rightarrow -i$ as $z \rightarrow \infty$. Roughly speaking, the invariant dynamical rays $R_{(a,b)}(0)$ and $R_{(a,b)}(1/2)$ tend to $+i\infty$ and $-i\infty$ respectively as the potential tends to infinity. By symmetry with respect to the real line, the 2-periodic rays $R_{(a,b)}(1/4)$ and $R_{(a,b)}(3/4)$ are contained in the real line. Also note that $-g_{a,b}(-z) = g_{a,-b}(z)$. Nonetheless, to define straightening maps consistently, we need to distinguish $g_{a,b}$ and $g_{a,-b}$ as they have different rational laminations (with respect to our normalized Böttcher coordinates). We will denote the connectedness locus of \mathcal{G} by $\mathcal{C}(\mathcal{G})$.

Remark. The parameter space of the family $\tilde{\mathcal{G}} = \{\tilde{g}_{a,b}(z) = z^3 + 3a^2z + b, a \geq 0, b \in \mathbb{R}\}$ of real cubic polynomials also contains tricorn-like sets. However, the definition of tricorn-like sets, the proof of umbilical cord wiggling, and discontinuity of straightening for the family $\tilde{\mathcal{G}}$ are completely analogous to those for the family \mathcal{G} . Hence we work out the details only for the family \mathcal{G} .

8.1. The Hyperbolic Component of Period One. Before studying renormalizations, we give an explicit description of the hyperbolic component of period one of \mathcal{G} .

Let,

$$\text{Per}_p(\lambda) = \{(a, b) : g_{a,b} \text{ has a periodic point of period } p \text{ with multiplier } \lambda\}.$$

It is easy to see that each $g_{a,b}$ in our family has exactly one real fixed point x , and exactly two non-real fixed points.

Lemma 8.1 (Indifferent Fixed Points). *If $g_{a,b}$ has an indifferent fixed point, then it is real and its multiplier is -1 .*

Proof. First observe that there is no parabolic fixed point of multiplier one. In fact, fixed points are the roots of

$$g_{a,b}(z) - z = -z^3 - (3a^2 + 1)z + b.$$

If a fixed point is parabolic of multiplier one, then the discriminant of $g_{a,b}(z) - z$ would vanish: i.e.

$$27b^2 = -4(3a^2 + 1)^3.$$

Clearly, there is no real (a, b) satisfying this equation, and hence $g_{a,b}$ cannot have a parabolic fixed point of multiplier 1.

²This parametrization has the advantage that the critical orbits are complex conjugate.

Assume an indifferent fixed point y is not real. Then there is also a symmetric (non-real) indifferent fixed point \bar{y} . Since we have already seen that the common multiplier of y and \bar{y} is not equal to 1, the invariant external rays (i.e., of angles 0 and $1/2$) cannot land at y and \bar{y} . Therefore, those rays must land at the other fixed point x on the real axis.

The critical points $\pm ai$ of $g_{a,b}$ are on the imaginary axis. Therefore, $g_{a,b}$ is monotone decreasing and $g_{a,b}^{\circ 2}$ is monotone increasing on \mathbb{R} . If $K(g_{a,b}) \cap \mathbb{R}$ contains an interval, then there must be a non-repelling fixed point for $g_{a,b}$ on \mathbb{R} . This is impossible because we already have two non-repelling cycles (in fact, fixed points). Therefore, we have $K(g_{a,b}) \cap \mathbb{R} = \{x\}$. By symmetry, the external rays at angles $1/4$ and $3/4$ are contained in the real line (these rays have period 2). Hence they both land at x . Therefore, x is the landing point of periodic external rays of different periods, which is a contradiction.

Therefore, any indifferent fixed point must be real, and since its multiplier is also real and not equal to one, it is equal to -1 . \square

Therefore the set of parameters with indifferent fixed points is equal to $\text{Per}_1(-1)$:

$$(1) \quad \text{Per}_1(-1) = \{(a, b) \in \mathbb{R}_{\geq 0} \times \mathbb{R} : 4(3a^2 - 1)(3a^2 + 2)^2 + 27b^2 = 0\}.$$

Let $\Phi_1(a) = -4(3a^2 - 1)(3a^2 + 2)^2$. Therefore, the (unique) hyperbolic component with attracting fixed point is defined by

$$\mathcal{H}_1 = \{(a, b) \in \mathbb{R}^2 : a \in [0, \frac{1}{\sqrt{3}}), 27b^2 < \Phi_1(a)\}.$$

We end this subsection with a brief remark on the structure of the period two hyperbolic components of \mathcal{G} . As depicted in Figure 8, there are three types of hyperbolic components of period 2 in this parameter space. There is only one bitransitive hyperbolic component of period 2 having center at $(\frac{1}{\sqrt{2}}, 0)$, and this component touches the unique period 1 hyperbolic component along a part of $\text{Per}_1(-1)$. There are three disjoint-type hyperbolic components of period 2 that ‘bifurcate’ from the unique period 2 bitransitive hyperbolic component across sub-arcs of $\text{Per}_2(1)$. Furthermore, there are two adjacent-type hyperbolic components each of which touches the unique period 1 hyperbolic component along a sub-arc of $\text{Per}_1(-1)$, and a disjoint-type hyperbolic component of period 2 along a sub-arc of $\text{Per}_2(1)$. The proofs of these facts are highly algebraic, and we omit them here.

8.2. Centers of Bitransitive Components. Since we will be concerned with renormalizations based at bitransitive hyperbolic components of the family \mathcal{G} , we need to take a closer look at the dynamics of the centers of such components. Throughout this subsection, we assume that (a_0, b_0) is the center of a bitransitive hyperbolic component period $2n$ (necessarily even due to the symmetry with respect to the real line); i.e. $g_{a_0, b_0}^{\circ 2n}(\pm ia_0) = \pm ia_0$. Let k be the smallest positive integer such that $g_{a_0, b_0}^{\circ k}(ia_0) = -ia_0$.

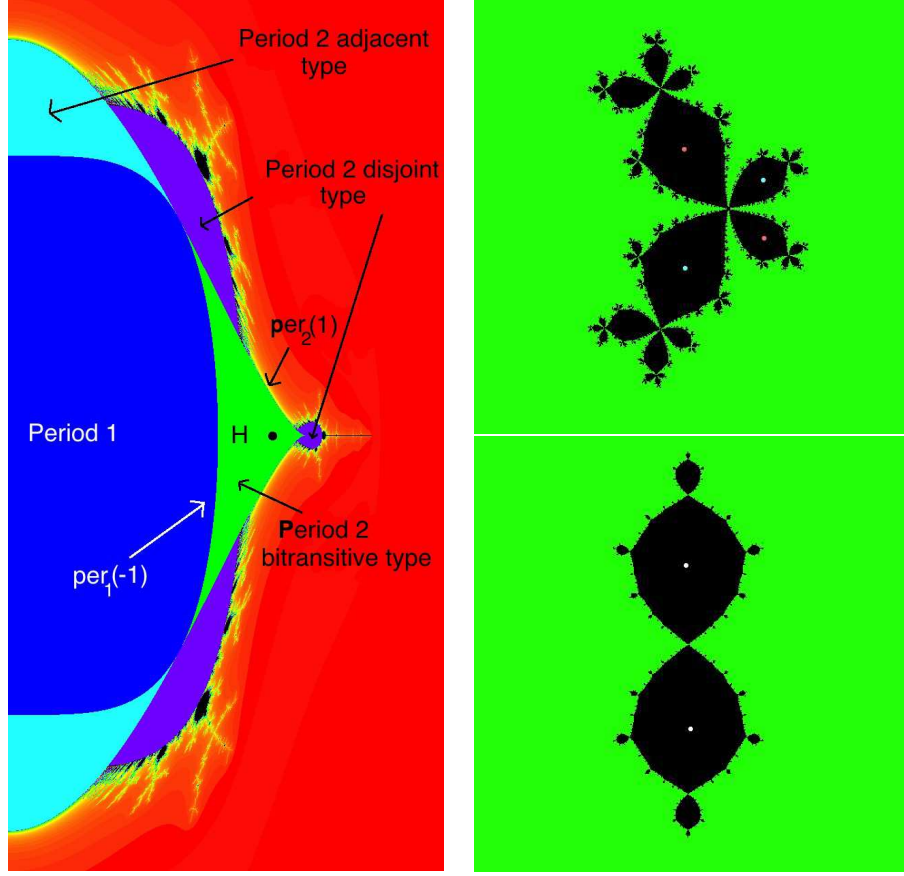


FIGURE 8. Left: A cartoon of the parameter space of the family \mathcal{G} highlighting the period 1 and period 2 components. For $(a_0, b_0) = (\frac{1}{\sqrt{2}}, 0)$, the corresponding renormalization locus $\mathcal{R}(a_0, b_0)$ fails to be compact precisely along a sub-arc of $\text{Per}_1(-1) \cup \text{Per}_2(1)$. Top right: The filled-in Julia set of the center of a disjoint-type hyperbolic component of period 2 with the two distinct super-attracting 2-cycles marked. Bottom right: The filled-in Julia set of the center $(\frac{1}{\sqrt{2}}, 0)$ of the unique bitransitive-type hyperbolic component of period 2 with its super-attracting 2-cycle marked. The two periodic bounded Fatou components touch at the origin, so $g_{\frac{1}{\sqrt{2}}, 0}$ is not primitive.

Then $\iota \circ g_{a_0, b_0}^{\circ k}(ia_0) = ia_0$; i.e. $g_{a_0, b_0}^{\circ k} \circ \iota(ia_0) = ia_0$, and $g_{a_0, b_0}^{\circ k}(-ia_0) = ia_0$. Therefore, $g_{a_0, b_0}^{\circ 2k}(ia_0) = ia_0$. This implies that $k = n$.

Lemma 8.2. *$K(g_{a_0, b_0})$ intersects \mathbb{R} at a single point, which is the unique real fixed point x of g_{a_0, b_0} .*

Proof. Since $K(g_{a_0, b_0})$ is connected, full, compact, and symmetric with respect to the real line, its intersection with the real line is either a singleton $\{x\}$, or an interval $[p, q]$ with $q > p$. We assume the latter case. Then p and q are the landing points of $R_f(1/4)$ and $R_f(3/4)$. So $\{p, q\}$ is a repelling 2-cycle (cannot be parabolic as both critical points are periodic). Since g_{a_0, b_0} is a real polynomial, and has no critical point on $[p, q]$, $g_{a_0, b_0} : [p, q] \rightarrow [p, q]$ is a strictly monotone map. But $g'_{a_0, b_0}(z) = -3(z^2 + a_0^2)$, which is negative for z in \mathbb{R} . Thus g_{a_0, b_0} is strictly decreasing, and $g_{a_0, b_0}^{\circ 2}$ is strictly increasing on $\{p, q\}$. As $\{p, q\}$ is a repelling 2-cycle, it follows that $[p, q]$ contains a non-repelling cycle of g_{a_0, b_0} . This is impossible because both critical points $\pm ia_0$ are periodic, and away from the real line. Hence, $\mathbb{R} \cap K(g_{a_0, b_0}) = \{x\}$. \square

In light of Theorem 5.4, it will be useful to know when the post-critically finite polynomial g_{a_0, b_0} is primitive. We answer this question in the following two lemmas.

Let us first discuss the special case when $n = 1$. The parameter $(a_0, b_0) = (1/\sqrt{2}, 0)$ is the center of the unique period 2 bitransitive hyperbolic component. More precisely, $g_{a_0, b_0}(ia_0) = -ia_0$, and $g_{a_0, b_0}(-ia_0) = ia_0$. Let U_1 and U_2 be the Fatou components of g_{a_0, b_0} containing ia_0 and $-ia_0$ respectively.

Lemma 8.3 (The $n = 1$ Case). *$\partial U_1 \cap \partial U_2 = \{0\}$. In particular, $g_{\frac{1}{\sqrt{2}}, 0}$ is not primitive.*

Proof. Observe that g_{a_0, b_0} commutes with the reflection with respect to $i\mathbb{R}$, hence $-\iota(K(g_{a_0, b_0})) = K(g_{a_0, b_0})$. Since $\pm ia_0 \in K(g_{a_0, b_0})$, and $K(g_{a_0, b_0})$ is connected, full, compact, it follows that $[-ia_0, ia_0] \subset K(g_{a_0, b_0})$. Since 0 is the unique fixed point of g_{a_0, b_0} on the real line, it follows by Lemma 8.2 that $\mathbb{R} \cap K(g_{a_0, b_0}) = \{0\}$. Since 0 is repelling, it belongs to the Julia set.

g_{a_0, b_0} has no critical point in $(-ia_0, ia_0)$, so g_{a_0, b_0} is strictly monotone there. Since $g'_{a_0, b_0}(0) = -3/2 < 0$, $g_{a_0, b_0}|_{[-ia_0, ia_0]}$ is strictly decreasing and $g_{a_0, b_0}^{\circ 2}|_{[-ia_0, ia_0]}$ is strictly increasing. It is now an easy exercise in interval dynamics to see that the $g_{a_0, b_0}^{\circ 2}$ -orbit of each point in $(0, ia_0)$ converges to the super-attracting point ia_0 , and hence $(0, ia_0] \subset U_1$. Since 0 is in the Julia set, it follows that $0 \in \partial U_1$.

A similar argument shows that $0 \in \partial U_2$. But the boundaries of two bounded Fatou components of a polynomial cannot intersect at more than one point. This proves that $\partial U_1 \cap \partial U_2 = \{0\}$ (compare bottom right of Figure 8). \square

Now let $n > 1$.

Lemma 8.4 (The $n > 1$ Case). *If n is larger than 1, then g_{a_0, b_0} is primitive.*

Proof. Using a Hubbard tree argument (compare [NS03, Lemma 3.4]), it is easy to see that every periodic bounded Fatou component of g_{a_0, b_0} has exactly one root. Let U_1 and U_2 be two distinct Fatou components of g_{a_0, b_0} with $U_1 \cap U_2 \neq \emptyset$. We can take iterated forward images to assume that ∂U_1

and ∂U_2 are periodic. Then the intersection $\partial U_1 \cap \partial U_2$ consists only of the unique common ‘root’ x of U_1 and U_2 .

Let U_1, U_2, \dots, U_r be all the periodic components touching at x . Since g_{a_0, b_0} commutes with ι and there is only one cycle of periodic component, it follows that $g_{a_0, b_0}^{\circ n}(U_j) = \iota(U_j)$, for $j = 1, 2, \dots, r$. $g_{a_0, b_0}^{\circ n}$ is a local orientation-preserving diffeomorphism from a neighborhood of x to a neighborhood of \bar{x} . But if $r \geq 3$, it would reverse the cyclic order of the Fatou components U_j touching at x . Hence, $r \leq 2$; i.e. at most 2 periodic (bounded) Fatou components can touch at x . This implies that x has period n , and $g_{a_0, b_0}^{\circ n}(U_1) = U_2$. Hence $U_2 = \iota(U_1)$. The upshot of this is that ∂U_1 and $\iota(\partial U_1)$ intersect at x . But by Lemma 8.2, x must be the unique real fixed point of g_{a_0, b_0} . This contradicts the assumption that $n > 1$.

Therefore, all bounded Fatou components of g_{a_0, b_0} have disjoint closures, and g_{a_0, b_0} is primitive. \square

8.3. Renormalizations of Bitransitive Components, and Tricorn-like Sets. Let (a_0, b_0) be the center of a bitransitive hyperbolic component of period $2n$; i.e. $g_{a_0, b_0}^{\circ n}(ia_0) = -ia_0$ and $g_{a_0, b_0}^{\circ n}(-ia_0) = ia_0$ for some $n \geq 1$. Then there exists a neighborhood U_0 of the closure of the Fatou component containing ia_0 such that U_0 is compactly contained in $\iota \circ g_{a_0, b_0}^{\circ n}(U_0)$ with $\iota \circ g_{a_0, b_0}^{\circ n} : U_0 \rightarrow (\iota \circ g_{a_0, b_0}^{\circ n})(U_0)$ proper (compare Figure 2). Since $\iota \circ g_{a_0, b_0}^{\circ n}$ is an antiholomorphic map of degree 2, we have an anti-polynomial-like map of degree 2 (with a connected filled-in Julia set) defined on U . The straightening Theorem 5.1 now yields a quadratic antiholomorphic polynomial (with a connected filled-in Julia set) that is hybrid equivalent to $(\iota \circ g_{a_0, b_0}^{\circ n})|_U$. One can continue to perform this renormalization procedure as the real cubic polynomial $g_{a, b}$ moves in the parameter space, and this defines a map from a suitable region in the parameter plane of real cubic polynomials to the tricorn.

More precisely, let $\lambda_{a, b}$ be the rational lamination of $g_{a, b}$. Define the combinatorial renormalization locus to be

$$\mathcal{C}(a_0, b_0) = \{(a, b) \in \mathbb{R}^2 : a \geq 0, \lambda_{a, b} \supset \lambda_{a_0, b_0}\},$$

and the renormalization locus to be

$$\mathcal{R}(a_0, b_0) = \{(a, b) \in \mathcal{C}(a_0, b_0) : \exists U', U \text{ such that } \iota \circ g_{a, b}^{\circ n} : U' \rightarrow U \text{ is anti-polynomial-like of degree 2 with a connected filled-in Julia set}\}.$$

Using Theorem 5.1, for each $(a, b) \in \mathcal{R}(a_0, b_0)$, we can straighten $\iota \circ g_{a, b}^{\circ n} : U' \rightarrow U$ to obtain a quadratic antiholomorphic polynomial f_c . This defines the straightening map

$$\begin{aligned} \chi_{a_0, b_0} : \mathcal{R}(a_0, b_0) &\rightarrow \mathcal{M}_2^* \\ (a, b) &\mapsto c. \end{aligned}$$

The proof of the fact that our definition of χ_{a_0, b_0} agrees with the general definition of straightening maps [IK12] goes as in Section 5.

At this point, we have to distinguish between the cases $n = 1$ and $n > 1$. By Lemma 8.4, g_{a_0, b_0} is primitive whenever $n > 1$. Therefore, the analogues of Theorem 5.2, Theorem 5.3, and Theorem 5.4 hold when $n > 1$.

On the other hand, when $n = 1$, Lemma 8.3 tells that g_{a_0, b_0} is not primitive. Hence in this case, $\mathcal{R}(a_0, b_0)$ is a proper non-compact subset of $\mathcal{C}(a_0, b_0)$. However, Theorem 5.3 implies the following:

Proposition 8.5. *The image of the straightening map χ_{a_0, b_0} contains the hyperbolicity locus in $\text{int } \mathcal{M}^*$.*

Indeed, the proposition holds for any straightening map χ_{a_0, b_0} for anti-holomorphic renormalizations in the family $\{g_{a, b}\}$.

Proof. As in Section 5, we complexify the family and consider the straightening map defined there.

Let $\text{Poly}(3) = \{P_{A, b}(z) = z^3 - 3Az + b; (A, b) \in \mathbb{C}^2\}$ denote the complex cubic family. Observe that $f_{a, b} = P_{a^2, bi}$. Let $\tilde{\chi}_{A_0, b_0 i} : \tilde{\mathcal{C}}(A_0, b_0 i) \rightarrow \mathcal{C}(2 \times 2)$ be the straightening map for $(A_0, b_0 i)$ -renormalization in $\text{Poly}(3)$, where $A_0 = a_0^2$.

Then by Theorem 5.3, for any hyperbolic parameter $c \in \mathcal{M}^*$, there exists some $(A, b) \in \mathbb{C}^2$ such that $\tilde{\chi}_{A_0, b_0 i}(A, b)$ is defined and equal to $(Q_{c_0}, Q_{c_1}) = (Q_{\bar{c}}, Q_c)$. Hence if A is real and b is purely imaginary, then it follows that $\chi_{a_0, b_0}(a, \frac{b}{i}) = c$, where $a = \sqrt{A}$.

In fact, let us consider $P(z) = P_{A, b}(z) = z^3 - 3Az + b$. Since f_{a_0, b_0} is symmetric on the imaginary axis, $\varphi(P(\varphi(z))) = P_{\bar{A}, -\bar{b}}$ is also $(A_0, b_0 i)$ -renormalizable where $\varphi(z) = -\bar{z}$ is the reflection with respect to the imaginary axis, and since φ exchanges the critical points $\pm a_0$ for $P_{A_0, b_0 i}$, we have

$$\tilde{\chi}_{A_0, b_0 i}(\bar{A}, -\bar{b}) = (\overline{Q_{c_1}(\bar{z})}, \overline{Q_{c_0}(\bar{z})}) = (Q_{\bar{c}}, Q_c) = \tilde{\chi}_{A_0, b_0 i}(A, b).$$

Therefore, by Theorem 5.2, $(\bar{A}, -\bar{b}) = (A, b)$, equivalently, $A \in \mathbb{R}$ and $b \in i\mathbb{R}$. \square

With these preliminary results at our disposal, we can now set up the foundation for the key technical theorem (of this section) to the effect that all ‘umbilical cords’ away from the line $\{b = 0\}$ ‘wiggle’. Let H_1, H_2, H_3 be the hyperbolic components of period 3 of \mathcal{M}_2^* (by [MNS15, Theorem 1.3], there are only 3 of them). Since $\chi_{a_0, b_0}(\mathcal{R}(a_0, b_0))$ contains the hyperbolic components of \mathcal{M}_2^* (by Proposition 8.5), there exists $i \in \{1, 2, 3\}$ such that $H' := \chi_{a_0, b_0}^{-1}(H_i)$ does not intersect the line $\{b = 0\}$ (but is contained in $\mathcal{R}(a_0, b_0)$). Then $\partial H'$ consists of 3 parabolic cusps and 3 parabolic arcs which are parametrized by the critical Ecalle height. Let (\tilde{a}, \tilde{b}) be the critical Ecalle height 0 parameter on the root arc (such that the unique parabolic cycle disconnects the Julia set) of $\partial H'$. Let U be the unique Fatou component of $g_{\tilde{a}, \tilde{b}}$ containing the critical point ai , and \tilde{z} be the unique parabolic periodic point of $g_{\tilde{a}, \tilde{b}}$ on ∂U . Since $g_{\tilde{a}, \tilde{b}}$ commutes with ι , $\iota(U)$ is the unique Fatou

component of $g_{\tilde{a}, \tilde{b}}$ containing the critical point $-ai$, and \tilde{z}^* is the unique parabolic periodic point of $g_{\tilde{a}, \tilde{b}}$ on $\partial\iota(U)$. We will show that if the ‘umbilical cord’ of H' lands, then the two polynomial-like restrictions of $g_{\tilde{a}, \tilde{b}}^{\circ 2n}$ in some neighborhoods of \overline{U} and $\overline{\iota(U)}$ (respectively) are conformally conjugate.

Lemma 8.6. *If there exists a path $p : [0, \delta] \rightarrow \mathbb{R}^2$ such that $p(0) = (\tilde{a}, \tilde{b})$, and $p((0, \delta]) \subset \mathcal{C}(\mathcal{G}) \setminus \overline{H'}$, then the two polynomial-like restrictions of $g_{\tilde{a}, \tilde{b}}^{\circ 2n}$ in some neighborhoods of \overline{U} , and $\overline{\iota(U)}$ respectively are conformally conjugate.*

Proof. The proof is essentially the same as for the maps f_c , so we only give a sketch.

Applying the parabolic implosion techniques [IM16, Lemma 2.5], one shows that the existence of such a path p would imply that the repelling equator at \tilde{z} is contained in a loose parabolic tree of $g_{\tilde{a}, \tilde{b}}$. Note that the proof of this fact in Lemma 3.2 made use of the parameter rays of the multicorns. However, one can circumvent that by the following argument. If the repelling equator at \tilde{z} is not contained in a loose parabolic tree of $g_{\tilde{a}, \tilde{b}}$, then there would exist dynamical rays (in the dynamical plane of $g_{\tilde{a}, \tilde{b}}$) traversing an interval of outgoing Ecalle heights $[-x, x]$ with $x > 0$. This would remain true after perturbation. For $s > 0$, the critical orbits of $g_{p(s)}$ “transit” from the incoming Ecalle cylinder to the outgoing cylinder (the two critical orbits are related by the conjugacy ι); as $s \downarrow 0$, the image of the critical orbits in the outgoing Ecalle cylinder has (outgoing) Ecalle height tending to 0, while the phase tends to $-\infty$ [IM16, Lemma 2.5]. Therefore, there exists $s \in (0, \delta)$ arbitrarily close to 0 for which the critical orbit(s), projected into the incoming cylinder, and sent by the transit map to the outgoing cylinder, land(s) on the projection of some dynamical ray that crosses the equator. But in the dynamics of $g_{p(s)}$, this means that the critical orbits lie in the basin of infinity, i.e. such a parameter $p(s)$ lies outside $\mathcal{C}(\mathcal{G})$. This contradicts our assumption that $p((0, \delta]) \subset \mathcal{C}(\mathcal{G}) \setminus \overline{H'}$.

If the repelling equator at \tilde{z} is contained in a loose parabolic tree of $g_{\tilde{a}, \tilde{b}}$, then one can argue as in Lemma 3.3 to conclude that there exists a real-analytic arc Γ , which is invariant under $g_{\tilde{a}, \tilde{b}}^{\circ 2n}$, and passes through \tilde{z} . This implies that $g_{\tilde{a}, \tilde{b}}^{\circ 2n}|_{B(\tilde{z}, \varepsilon)}$ (for ε sufficiently small) is conformally conjugate to a real parabolic germ.

One can carry out the same argument with the parabolic periodic point \tilde{z}^* (as in Lemma 3.3) to conclude that $g_{\tilde{a}, \tilde{b}}^{\circ 2n}|_{B(\tilde{z}^*, \varepsilon)}$ is conformally conjugate to the same real parabolic germ as the one constructed in the previous paragraph. Thus, we have a local biholomorphism η between the parabolic germs $g_{\tilde{a}, \tilde{b}}^{\circ 2n}|_{B(\tilde{z}, \varepsilon)}$ and $g_{\tilde{a}, \tilde{b}}^{\circ 2n}|_{B(\tilde{z}^*, \varepsilon)}$ such that η maps $g_{\tilde{a}, \tilde{b}}^{\circ 2rn}(ai)$ to $g_{\tilde{a}, \tilde{b}}^{\circ 2rn}(-ai)$, for r large enough so that $g_{\tilde{a}, \tilde{b}}^{\circ 2rn}(ai)$ lies in the domain of definition of η .

Observe that $g_{\tilde{a},\tilde{b}}^{\circ 2n}|_U$ and $g_{\tilde{a},\tilde{b}}^{\circ 2n}|_{\iota(U)}$ are conformally conjugate (by the condition of having critical Ecalle height 0). Hence, we can extend η to a conformal conjugacy between the polynomial-like restrictions of $g_{\tilde{a},\tilde{b}}^{\circ 2n}$ in some neighborhoods of \overline{U} and $\overline{\iota(U)}$ respectively (arguing as in Lemma 4.2 and Lemma 4.3). \square

Remark. We would like to emphasize, albeit at the risk of being pedantic, that the germ conjugacy η extends to the closure of the basins only because it respects the dynamics on the critical orbits.

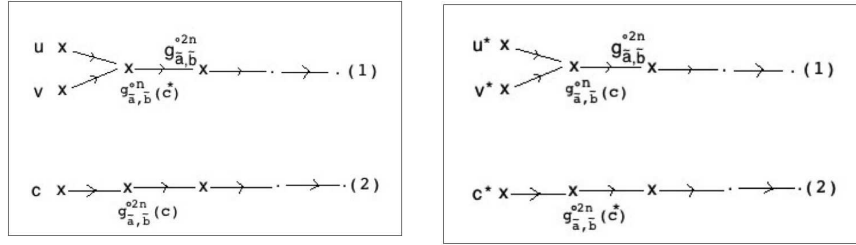


FIGURE 9. Left: The dynamics on the two dynamically marked critical orbits of $g_{\tilde{a},\tilde{b}}^{\circ 2n}|_U$. Right: The dynamics on the two dynamically marked critical orbits of $g_{\tilde{a},\tilde{b}}^{\circ 2n}|_{\iota(U)}$. Observe that $g_{\tilde{a},\tilde{b}}^{\circ n}$ swaps the two dynamically marked critical orbits.

$g_{\tilde{a},\tilde{b}}^{\circ 2n}$ has three critical points $u, v, c = ai$ in the Fatou component U , such that $g_{\tilde{a},\tilde{b}}^{\circ 2n}(u) = g_{\tilde{a},\tilde{b}}^{\circ 2n}(v) = g_{\tilde{a},\tilde{b}}^{\circ n}(-ai) = g_{\tilde{a},\tilde{b}}^{\circ n}(c^*)$. Therefore $g_{\tilde{a},\tilde{b}}^{\circ 2n}$ has two infinite critical orbits in U , and these two critical orbits are dynamically different. By our construction, η maps (the tail of) each of the two (dynamically distinct) critical orbits of $g_{\tilde{a},\tilde{b}}^{\circ 2n}|_U$ to the (tail of the) corresponding critical orbit of $g_{\tilde{a},\tilde{b}}^{\circ 2n}|_{\iota(U)}$. In fact, the parabolic germs $g_{\tilde{a},\tilde{b}}^{\circ 2n}|_{B(\tilde{z},\varepsilon)}$ and $g_{\tilde{a},\tilde{b}}^{\circ 2n}|_{B(\tilde{z}^*,\varepsilon)}$ are always conformally conjugate by $g_{\tilde{a},\tilde{b}}^{\circ n}$; but this local conjugacy exchanges the two dynamically marked post-critical orbits (compare Figure 9), which have different topological dynamics, and hence this local conjugacy has no chance of being extended to the entire parabolic basin.

Theorem 8.7 (Umbilical Cord Wiggling in Real Cubics). *There does not exist a path $p : [0, \delta] \rightarrow \mathbb{R}^2$ such that $p(0) = (\tilde{a}, \tilde{b})$, and $p((0, \delta]) \subset \mathcal{C}(\mathcal{G}) \setminus \overline{H'}$.*

Proof. We have already showed that the existence of the required path implies that the polynomial-like restrictions $g_{\tilde{a},\tilde{b}}^{\circ 2n} : U' \rightarrow g_{\tilde{a},\tilde{b}}^{\circ 2n}(U')$ (where U' is a neighborhood of \overline{U}), and $g_{\tilde{a},\tilde{b}}^{\circ 2n} : \iota(U') \rightarrow g_{\tilde{a},\tilde{b}}^{\circ 2n}(\iota(U'))$ (where $\iota(U')$ is a neighborhood of $\overline{\iota(U)}$) are conformally conjugate. Applying Theorem [Ino11,

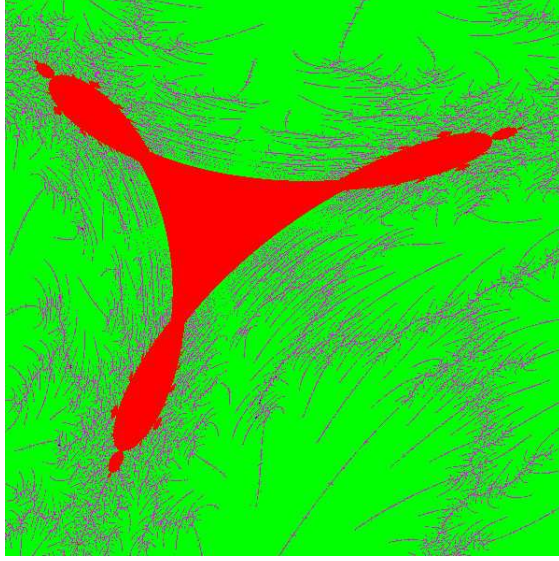


FIGURE 10. Wiggling of an umbilical cord for a tricorn-like set in the real cubic locus.

Theorem 1] to this situation, and arguing as in Lemma 4.4, we obtain the existence of polynomials p , p_1 and p_2 such that

$$g_{\tilde{a}, \tilde{b}}^{\circ 2n} \circ p_1 = p_1 \circ p, \quad g_{\tilde{a}, \tilde{b}}^{\circ 2n} \circ p_2 = p_2 \circ p, \quad \text{and} \quad \deg p_1 = \deg p_2.$$

Moreover, by Theorem [Ino11, Theorem 1], p has a polynomial-like restriction $p : V \rightarrow p(V)$ which is conformally conjugate to $g_{\tilde{a}, \tilde{b}}^{\circ 2n} : U' \rightarrow g_{\tilde{a}, \tilde{b}}^{\circ 2n}(U')$ by p_1 , and to $g_{\tilde{a}, \tilde{b}}^{\circ 2n} : \iota(U') \rightarrow g_{\tilde{a}, \tilde{b}}^{\circ 2n}(\iota(U'))$ by p_2 . We now consider two cases.

Case 1: $\deg(p_1) = \deg(p_2) = 1$. Set $p_3 := p_1 \circ p_2^{-1}$. Then p_3 is an affine map commuting with $g_{\tilde{a}, \tilde{b}}^{\circ 2n}$, and conjugating the two polynomial-like restrictions of $g_{\tilde{a}, \tilde{b}}^{\circ 2n}$ under consideration. Clearly, $p_3 \neq \text{id}$. An easy computation (using the fact that $g_{\tilde{a}, \tilde{b}}^{\circ 2n}$ is a centered real polynomial) now shows that $p_3(z) = -z$, and hence $\tilde{b} = 0$.

Case 2: $\deg(p_1) = \deg(p_2) = k > 1$. The arguments employed in this case are morally similar to the last step in the proof of Theorem 1.2. We will first prove by contradiction that $\gcd(\deg g_{\tilde{a}, \tilde{b}}^{\circ 2n}, \deg p_1) > 1$. To do this, let $\gcd(\deg g_{\tilde{a}, \tilde{b}}^{\circ 2n}, \deg p_1) = 1$. Now we can apply Theorem 4.6 to our situation. Since $g_{\tilde{a}, \tilde{b}}^{\circ 2n}$ is parabolic, it is neither a power map, nor a Chebyshev polynomial. Hence, there exists some non-constant polynomial P such that $g_{\tilde{a}, \tilde{b}}^{\circ 2n}$ is affinely conjugate to the polynomial $h(z) := z^r(P(z))^k$, and $p_1(z) = z^k$ (up to affine conjugacy). If $r \geq 2$, then $h(z)$ has a super-attracting fixed

point at 0. But $g_{\tilde{a},\tilde{b}}^{\circ 2n}$, which is affinely conjugate to $h(z)$, has no super-attracting fixed point. Hence, $r = 0$ or 1 . By degree consideration, we have $3^{2n} = r + ks$, where $\deg P = s$. The assumption $\gcd(\deg g_{\tilde{a},\tilde{b}}^{\circ 2n}, \deg p_1) = \gcd(3^{2n}, k) = 1$ implies that $r = 1$, i.e. $h(z) = z(P(z))^k$. Now the fixed point 0 for h satisfies $h^{-1}(0) = \{0\} \cup P^{-1}(0)$, and any point in $P^{-1}(0)$ has a local mapping degree k under h . The same must hold for the affinely conjugate polynomial $g_{\tilde{a},\tilde{b}}^{\circ 2n}$: there exists a fixed point (say x) for $g_{\tilde{a},\tilde{b}}^{\circ 2n}$ such that any point in $(g_{\tilde{a},\tilde{b}}^{\circ 2n})^{-1}(x)$ has mapping degree k (possibly) except for x ; in particular, all points in $(g_{\tilde{a},\tilde{b}}^{\circ 2n})^{-1}(x) \setminus \{x\}$ are critical points for $g_{\tilde{a},\tilde{b}}^{\circ 2n}$ (since $k > 1$). But this implies that $g_{\tilde{a},\tilde{b}}^{\circ 2n}$ has a finite critical orbit, which is a contradiction to the fact that all critical orbits of $g_{\tilde{a},\tilde{b}}^{\circ 2n}$ non-trivially converge to parabolic fixed points.

Now applying Engstrom's theorem [Eng41], we obtain the existence of polynomials (of degree at least two) α, β such that

$$g_{\tilde{a},\tilde{b}}^{\circ 2n} = \alpha \circ \beta, \quad p = \beta \circ \alpha.$$

Since $g_{\tilde{a},\tilde{b}}$ is a prime polynomial under composition (since its degree is a prime number), it follows that $p = g_{\tilde{a},\tilde{b}}^{\circ 2n}$. Therefore, p_1 commutes with $g_{\tilde{a},\tilde{b}}^{\circ 2n}$. As $g_{\tilde{a},\tilde{b}}^{\circ 2n}$ is neither a power map nor a Chebyshev polynomial, $p_1 = g_{\tilde{a},\tilde{b}}^{\circ k_1}$, for some $k_1 \in \mathbb{N}$ (up to affine conjugacy). The same is true for p_2 as well; i.e. $p_2 = g_{\tilde{a},\tilde{b}}^{\circ k_2}$ (up to affine conjugacy).

Therefore, there is a polynomial-like restriction of $p = g_{\tilde{a},\tilde{b}}^{\circ 2n} : V \rightarrow g_{\tilde{a},\tilde{b}}^{\circ 2n}(V)$, which is conformally conjugate to $g_{\tilde{a},\tilde{b}}^{\circ 2n} : U' \rightarrow g_{\tilde{a},\tilde{b}}^{\circ 2n}(U')$ by $p_1 = g_{\tilde{a},\tilde{b}}^{\circ k_1}$, and to $g_{\tilde{a},\tilde{b}}^{\circ 2n} : \iota(U') \rightarrow g_{\tilde{a},\tilde{b}}^{\circ 2n}(\iota(U'))$ by $p_2 = g_{\tilde{a},\tilde{b}}^{\circ k_2}$. But the dynamical configuration implies that this is impossible (since there is only one parabolic cycle, and the unique cycle of immediate parabolic basins contains two critical points of $g_{\tilde{a},\tilde{b}}$, either p_1 or p_2 must have a critical point in their corresponding conjugating domain).

Therefore, we have showed that the existence of such a path p would imply that $\tilde{b} = 0$. But this contradicts our assumption that H' does not intersect the line $\{b = 0\}$. This completes the proof of the theorem. \square

Using Theorem 8.7, we can now proceed as in Theorem 1.1 to prove that the straightening map $\chi_{a_0,b_0} : \mathcal{R}(a_0,b_0) \rightarrow \mathcal{M}_2^*$ is discontinuous.

Proof of Theorem 1.3. We will stick to the terminologies used throughout this section. We will assume that the map $\chi_{a_0,b_0} : \mathcal{R}(a_0,b_0) \rightarrow \mathcal{M}_2^*$ is continuous, and arrive at a contradiction. Due to technical reasons, we will split the proof in two cases.

Case 1: $n > 1$. We have observed that when n is larger than one, $\mathcal{R}(a_0, b_0)$ is compact. Moreover, the map χ_{a_0, b_0} is injective. Since an injective continuous map from a compact topological space onto a Hausdorff topological space is a homeomorphism, it follows that χ_{a_0, b_0} is a homeomorphism from $\mathcal{R}(a_0, b_0)$ onto its range (we do not claim that $\chi_{a_0, b_0}(\mathcal{R}(a_0, b_0)) = \mathcal{M}_2^*$). In particular, $\chi_{a_0, b_0}(\mathcal{R}(a_0, b_0))$ is closed. Since real hyperbolic quadratic polynomials are dense in $\mathbb{R} \cap \mathcal{M}_2^*$ [GŚ97, AKLS09] (the tricorn and the Mandelbrot set agree on the real line), it follows from Theorem 5.3, and the 3-fold rotational symmetry of the tricorn that $(\mathbb{R} \cup \omega\mathbb{R} \cup \omega^2\mathbb{R}) \cap \mathcal{M}_2^* \subset \chi_{a_0, b_0}(\mathcal{R}(a_0, b_0))$.

Recall that there exists a piece γ of $\omega^i\mathbb{R} \cap \mathcal{M}_2^*$ that lies outside of $\overline{H_i}$ (for some $i \in \{1, 2, 3\}$, where H_1, H_2, H_3 are the hyperbolic components of period 3 of \mathcal{M}_2^*), and lands at the critical Ecalle height 0 parameter on the root parabolic arc of ∂H_i . By our assumption, χ_{a_0, b_0} is a homeomorphism; and hence the curve $\chi_{a_0, b_0}^{-1}(\gamma)$ lies in the exterior of $\overline{H'}$, and lands at the critical Ecalle height 0 parameter on the root arc of $\partial H'$ (critical Ecalle heights are preserved by hybrid equivalences). But this contradicts Theorem 8.7, and proves the theorem for $n > 1$.

Case 2: $n = 1$. Finally we look at $(a_0, b_0) = (\frac{1}{\sqrt{2}}, 0)$. Note that since g_{a_0, b_0} is not primitive in this case, $\mathcal{R}(a_0, b_0)$ is not compact. So we cannot use the arguments of Case 1 directly, and we have to work harder to demonstrate that the image of the straightening map contains a suitable interval of the real line.

In the dynamical plane of g_{a_0, b_0} , the real line consists of two external rays (at angles $1/4$ and $3/4$) as well as their common landing point 0, which is the unique real fixed point of g_{a_0, b_0} . Recall that the rational lamination of every polynomial in $\mathcal{R}(a_0, b_0)$ is stronger than that of g_{a_0, b_0} , and the dynamical $1/4$ and $3/4$ -rays are always contained in the real line. Therefore in the dynamical plane of every $(a, b) \in \mathcal{R}(a_0, b_0)$, the real line consists of two external rays (at angles $1/4$ and $3/4$) as well as their common landing point which is repelling. In order to obtain a period 1 renormalization for any polynomial in $\mathcal{R}(a_0, b_0)$, one simply has to perform a standard Yoccoz puzzle construction starting with the $1/4$ and $3/4$ rays, and then thicken the depth 1 puzzle (thickening essentially yields compact containment of the domain of the polynomial-like map in its range). Now, the only possibility of having a non-renormalizable map as a limit of maps in $\mathcal{R}(a_0, b_0)$ is if the dynamical $1/4$ and $3/4$ -rays land at parabolic points. This can happen in two different ways. If these two rays land at a common parabolic point (since such a parabolic fixed point would have two petals, it would prohibit the thickening procedure), then by Lemma 8.1, the multiplier of the parabolic fixed point must be -1 . On the other hand, if the dynamical $1/4$ and $3/4$ -rays land at two distinct parabolic points, then those parabolic points would form a 2-cycle with multiplier $+1$ (the conclusion about the multiplier

follows from the fact that the first return map fixes each dynamical ray). Therefore, $\overline{\mathcal{R}(a_0, b_0)} \setminus \mathcal{R}(a_0, b_0) \subset \text{Per}_1(-1) \cup \text{Per}_2(1)$.

As in the previous case, let H_i be a period 3 hyperbolic component of \mathcal{M}_2^* , and γ be a curve in $\omega^i \mathbb{R} \cap \mathcal{M}_2^*$ that lies outside of $\overline{H_i}$ and lands at the critical Ecalle height 0 parameter on the root parabolic arc of ∂H_i . For definiteness, let us choose $\gamma := \omega^i[-1.75, -1.25]$ (note that -1.75 is the root of the real period 3 ‘airplane’ component, and -1.25 is the root of the real period 4 component that bifurcates from the real period 2 ‘basilica’ component). Moreover, H_i is in the range of χ_{a_0, b_0} , and $H' = \chi_{a_0, b_0}^{-1}(H_i)$ does not intersect the line $\{b = 0\}$. To complete the proof of the theorem, it suffices to show that there is a compact set $K \subset \mathcal{R}(a_0, b_0)$ with $\overline{H_i} \cup \gamma \subset \chi_{a_0, b_0}(K)$. Indeed, if there exists such a set K , then $\chi_{a_0, b_0}|_K$ would be a homeomorphism (recall that χ_{a_0, b_0} is continuous by assumption). Therefore, the curve $\chi_{a_0, b_0}^{-1}(\gamma)$ would lie in the exterior of $\overline{H'}$, and land at the critical Ecalle height 0 parameter on the root arc of $\partial H'$. Once again, this contradicts Theorem 8.7, and completes the proof in the $n = 1$ case.

Let us now prove the existence of the required compact set K . Note that since $\overline{H'}$ is contained in the union of the hyperbolicity locus and $\text{Per}_6(1)$ of the family \mathcal{G} , it follows that $\overline{H'}$ is disjoint from $\text{Per}_1(-1) \cup \text{Per}_2(1)$. Hence $\overline{H'}$ is contained in a compact subset of $\mathcal{R}(a_0, b_0)$.

Let us denote the hyperbolic parameters of γ by γ^{hyp} . By Lemma 8.5, γ^{hyp} is contained in the range of χ . We will now show that $\chi^{-1}(\gamma^{\text{hyp}})$ does not accumulate on $\text{Per}_1(-1) \cup \text{Per}_2(1)$; i.e. $\chi^{-1}(\gamma^{\text{hyp}})$ is contained in a compact subset of $\mathcal{R}(a_0, b_0)$. To this end, observe that γ^{hyp} is contained in the 1/2-limb of a period 2 hyperbolic component of \mathcal{M}_2^* . So for each parameter on γ^{hyp} , two 4-periodic dynamical rays land at a common point of the corresponding Julia set. Hence for each parameter on $\chi^{-1}(\gamma^{\text{hyp}})$, two 4-periodic dynamical rays (e.g. at angles $61/80$ and $69/80$) land at a common point. If $\chi^{-1}(\gamma^{\text{hyp}})$ accumulates on some parameter on the parabolic curves $\text{Per}_1(-1) \cup \text{Per}_2(1)$, then the corresponding dynamical rays at angles $61/80$ and $69/80$ would have to co-land in the dynamical plane of that parameter. But there is no such landing relation for parameters on $\text{Per}_1(-1) \cup \text{Per}_2(1)$. This proves that $\chi^{-1}(\gamma^{\text{hyp}})$ is contained in a compact subset of $\mathcal{R}(a_0, b_0)$.

Combining the observations of the previous two paragraphs, we conclude that there is a compact subset K of $\mathcal{R}(a_0, b_0)$ that contains $\overline{H'} \cup \chi^{-1}(\gamma^{\text{hyp}})$. Since we assumed χ_{a_0, b_0} to be continuous, it follows that $\chi_{a_0, b_0}(K)$ is a closed set containing γ^{hyp} . But γ^{hyp} is dense in γ (by the density of hyperbolic quadratic polynomials in \mathbb{R}). Therefore, $\gamma \subset \chi_{a_0, b_0}(K)$. Therefore, K is the required compact subset of $\mathcal{R}(a_0, b_0)$ such that $\chi_{a_0, b_0}(K) \supset \overline{H_i} \cup \gamma$. \square

9. ARE ALL BABY MULTICORNS DYNAMICALLY DISTINCT?

We proved in Section 7 that for multicorns of even degree, the straightening map from a multicorn-like set based at an odd period (different from 1) hyperbolic component to the multicorn is discontinuous at infinitely many

parameters. However, we conjecture that there is a stronger form of discontinuity; i.e. any two baby multicorn-like sets (for multicorns of any degree) are dynamically different.

Recall that there are two important conformal invariants associated with every odd period non-cusp parabolic parameter; namely the critical Ecalle height, and the holomorphic fixed point index of the parabolic cycle. While critical Ecalle height is preserved by straightening maps, the parabolic fixed point index is in general not (since a hybrid equivalence does not necessarily preserve the external class of a polynomial-like map). In this section, we will prove that continuity of straightening maps would force the above two conformal invariants to be uniformly related along every parabolic arc. We will then look at some explicit tricorn-like sets, and use the above information to show that the straightening maps between these tricorn-like sets are discontinuous at infinitely many parameters on certain root parabolic arcs.

9.1. A Weak Version of The Conjecture. For $i = 1, 2$, let H_i be a hyperbolic component of odd period k_i of \mathcal{M}_d^* , \mathcal{C}_i be the root parabolic arc³ on ∂H_i , and $c_i : \mathbb{R} \rightarrow \mathcal{C}_i$ be the critical Ecalle height parametrization of \mathcal{C}_i . By Theorem 2.7 and Lemma 2.8, each end of \mathcal{C}_i intersects the boundary of a hyperbolic component of period $2k_i$ along a sub-arc of \mathcal{C}_i . Let $c_1(h_1)$ and $c_1(h_2)$ (with $h_1 > 0 > h_2$) be the parameters on \mathcal{C}_1 (respectively $c_2(h_3)$ and $c_2(h_4)$ be the parameters on \mathcal{C}_2 with $h_3 > 0 > h_4$) at which bifurcations from H_1 (respectively from H_2) to hyperbolic components of period $2k_1$ (respectively $2k_2$) start (compare Figure 11).

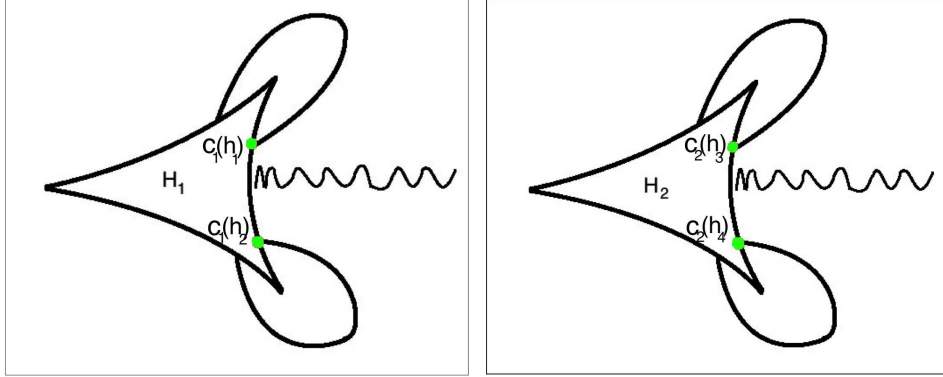


FIGURE 11. Left: Period doubling bifurcation from H_1 starting at parameters $c_1(h_1)$ and $c_1(h_2)$. Right: Period doubling bifurcation from H_2 starting at parameters $c_2(h_3)$ and $c_2(h_4)$.

Recall that critical Ecalle heights are preserved by hybrid equivalences, and hence by straightening maps. If the two multicorn-like sets based at

³One can also phrase the conjecture for co-root arcs, we work with root arcs for definiteness.

H_1 and H_2 were dynamically equivalent, then the ordered pairs (h_1, h_2) and (h_3, h_4) would be equal; i.e. $(h_1, h_2) = (h_3, h_4)$. However, these parameters are characterized by the condition of having residue fixed-point index (of their unique parabolic cycle) $+1$, and this being an ‘external conformal information’ for the polynomial-like restrictions (the hybrid class of a polynomial-like map does not determine the fixed point index), has no reason to be preserved by hybrid equivalences. We therefore make the following conjecture:

Conjecture 9.1 (Baby Multicorns are Dynamically Distinct - Weak Version). *If $\omega^i H_1 \neq H_2$ for $i = 1, 2, \dots, d+1$, then $(h_1, h_2) \neq (h_3, h_4)$.*

9.2. The Strong Version, and Supporting Evidences. By Theorem 6.3, straightening maps induce homeomorphisms between odd period hyperbolic components of \mathcal{M}_d^* . However, in this subsection, we will prove that the straightening map from any period 3 tricorn-like set to the original tricorn is discontinuous at all but possibly a discrete set of parameters on a sub-arc of the period 3 parabolic root arcs. We believe that this is also the general situation for the straightening map between any two (non-symmetric) tricorns-like sets.

We need to set up notations first. Let H be a hyperbolic component of odd period k , \mathcal{C} a parabolic arc of ∂H , $c_1 : \mathbb{R} \rightarrow \mathcal{C}$ be the critical Ecalle height parametrization of \mathcal{C} , and H' a hyperbolic component of period $2k$ bifurcating from H across \mathcal{C} . Any straightening map χ restricted to \overline{H} is a homeomorphism. Let $c_2 : \mathbb{R} \rightarrow \chi(\mathcal{C})$ be the critical Ecalle height parametrization of $\chi(\mathcal{C})$. Since χ preserves critical Ecalle heights, we have $c_2 = \chi \circ c_1$. For h sufficiently large, $c_1(h) \in \mathcal{C} \cap \partial H'$. Let the fixed point index of the unique parabolic cycle of $f_{c_1(h)}^{\circ 2}$ be τ . Consider a curve $\gamma : [0, 1] \rightarrow \overline{H'}$ with $\gamma(0) = c_1(h)$, and $\gamma((0, 1]) \subset H'$. For $t \neq 0$, $f_{\gamma(t)}^{\circ 2}$ has two distinct k -periodic attracting cycles (which are born out of the parabolic cycle) with multipliers $\lambda_{\gamma(t)}$ and $\overline{\lambda_{\gamma(t)}}$. Then,

$$(2) \quad \frac{1}{1 - \lambda_{\gamma(t)}} + \frac{1}{1 - \overline{\lambda_{\gamma(t)}}} \rightarrow \tau$$

as $t \downarrow 0$.

Continuity of the straightening map χ at $c_1(h)$ implies that $\lim_{t \downarrow 0} \chi(\gamma(t)) = c_2(h)$ (compare Figure 12). But the multipliers of attracting periodic orbits are preserved by χ . Therefore by the limiting relation (2), the fixed point index of the parabolic cycle of $f_{c_2(h)}^{\circ 2}$ is also τ . For any h in \mathbb{R} , let us denote the fixed point index of the unique parabolic cycle of $f_{c_1(h)}^{\circ 2}$ (respectively of $f_{c_2(h)}^{\circ 2}$) by $\text{ind}_{\mathcal{C}}(f_{c_1(h)}^{\circ 2})$ (respectively $\text{ind}_{\chi(\mathcal{C})}(f_{c_2(h)}^{\circ 2})$). Since $c_1(h)$ was an arbitrary parameter on $\mathcal{C} \cap \partial H'$, continuity of the straightening map on $\mathcal{C} \cap \partial H'$ would imply that the functions

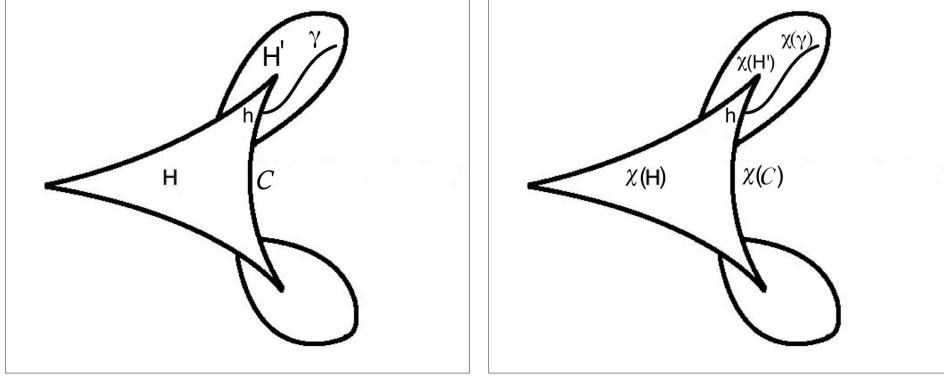


FIGURE 12. If χ was continuous at $c_1(h)$, then the fixed point indices of the parabolic cycles of $f_{c_1(h)}^{\circ 2}$ and $f_{c_2(h)}^{\circ 2}$ would be equal.

$$\begin{aligned} \text{ind}_{\mathcal{C}} : \mathbb{R} &\rightarrow \mathbb{R} \\ h &\mapsto \text{ind}_{\mathcal{C}}(f_{c_1(h)}^{\circ 2}). \end{aligned}$$

and

$$\begin{aligned} \text{ind}_{\chi(\mathcal{C})} : \mathbb{R} &\rightarrow \mathbb{R} \\ h &\mapsto \text{ind}_{\chi(\mathcal{C})}(f_{c_2(h)}^{\circ 2}). \end{aligned}$$

agree on an interval of positive length. Since these functions are real-analytic, the identity theorem implies that these two functions must agree everywhere!

This seems extremely unlikely to hold, but we do not know how to rule out this possibility in general. However, the advantage of the preceding analysis is that the ‘dynamical’ discontinuity of the straightening map would follow if we can prove that the functions $\text{ind}_{\mathcal{C}}$ and $\text{ind}_{\chi(\mathcal{C})}$ disagree at a single point. We apply these observations to show that the original tricorn, the period 3 tricorn-like sets in the original tricorn, and a period 4 tricorn-like set in the real cubic locus are all dynamically distinct. This is an indication of the fact that each tricorn-like set carries its own characteristic geometry, which distinguishes it from other tricorn-like sets.

More precisely, we have the following.

- (1) $(\mathcal{T}, \mathcal{C})$; where \mathcal{T} is the original tricorn \mathcal{M}_2^* , and \mathcal{C} is the period 1 parabolic arc intersecting the real line (of \mathcal{T}). The critical Ecalle height 0 map on \mathcal{C} is $f_{\frac{1}{4}}(z) = \bar{z}^2 + \frac{1}{4}$. The parabolic fixed point index of $f_{\frac{1}{4}}^{\circ 2}$ is $\frac{1}{2}$.
- (2) $(\mathcal{T}_1, \mathcal{C}_1)$; where \mathcal{T}_1 is the period 3 tricorn-like set (in the tricorn) intersecting the real line (equivalently, the renormalization locus $\mathcal{R}(\bar{z}^2 + c_0)$, where c_0 is the airplane parameter), and \mathcal{C}_1 is the unique period 3 root parabolic arc contained in \mathcal{T}_1 . The critical Ecalle height 0 map on \mathcal{C}_1 is $f_{-\frac{7}{4}}(z) = \bar{z}^2 - \frac{7}{4}$. The parabolic fixed point index of

$f_{-\frac{7}{4}}^{\circ 6}$ is $\frac{47}{98}$. One can compute this fixed point index as follows. The polynomial $z^2 - \frac{7}{4}$ has two distinct fixed points, and the corresponding multipliers λ_1 and λ_2 are related by the equations $\lambda_1 + \lambda_2 = 2$, and $\lambda_1 \lambda_2 = -7$. Let ξ be the fixed point index of the parabolic fixed points of the third iterate of $z^2 - \frac{7}{4}$. By the holomorphic fixed point formula (applied to the third iterate of $z^2 - \frac{7}{4}$),

$$3\xi + \frac{1}{1 - \lambda_1^3} + \frac{1}{1 - \lambda_2^3} = 0$$

A simple computation shows that $\xi = -\frac{2}{49}$. The parabolic fixed point index of $f_{-\frac{7}{4}}^{\circ 6}$ (which is the same as that of the sixth iterate of $z^2 - \frac{7}{4}$) is given by $\frac{1+\xi}{2} = \frac{47}{98}$ (compare [Mil06, Lemma 12.9]).

- (3) $(\mathcal{T}_2, \mathcal{C}_2)$; where \mathcal{T}_2 is the period 4 tricorn-like set (in the parameter space of the family \mathcal{G}) intersecting the line $\{b = 0\}$ (equivalently the renormalization locus $\mathcal{R}(-z^3 - \frac{3}{4}\sqrt{6+2\sqrt{17}}z)$), and \mathcal{C}_2 is the unique period 4 root parabolic arc contained in \mathcal{T}_2 . The critical Ecalle height 0 map on \mathcal{C}_2 is $g_{a,0}(z) = -z^3 - 2\sqrt{2}z$, where $a = (\frac{8}{9})^{\frac{1}{4}}$. The parabolic fixed point index of $g_{a,0}^{\circ 4}$ is $\frac{35}{72}$. As in the previous case, this can be computed by observing that $g_{a,0}^{\circ 4}$ is the second iterate of $-g_{a,0}^{\circ 2}$. Indeed, $-g_{a,0}^{\circ 2}$ (which has degree 9) has a real fixed point at 0 with multiplier $(-g_{a,0}^{\circ 2})'(0) = -8$, and four parabolic fixed points each of multiplicity two. It now follows from the holomorphic fixed point formula (applied to $-g_{a,0}^{\circ 2}$) that the fixed point index of each of the parabolic fixed points of $-g_{a,0}^{\circ 2}$ is $-\frac{1}{36}$. Hence, the parabolic fixed point index of its second iterate $g_{a,0}^{\circ 4}$ is $\frac{1-1/36}{2} = \frac{35}{72}$ (compare [Mil06, Lemma 12.9]).

Under straightening maps, the parabolic arcs \mathcal{C} , \mathcal{C}_1 , and \mathcal{C}_2 correspond to each other. But the parabolic fixed point indices of their critical Ecalle height 0 parameters are all distinct. Hence the functions $\text{ind}_{\mathcal{C}}$, $\text{ind}_{\mathcal{C}_1}$, and $\text{ind}_{\mathcal{C}_2}$ pairwise differ at all but possibly a discrete set of real numbers. Therefore the corresponding straightening maps are discontinuous at all but possibly a discrete set of parameters on a sub-arc of the corresponding parabolic arcs. We summarize these observations in the following proposition.

Proposition 9.2 (Dynamically Distinct Baby Tricorns). *The original tricorn, the period 3 tricorn-like sets in the original tricorn, and the period 4 tricorn-like set intersecting the line $\{b = 0\}$ in the real cubic locus \mathcal{G} are all dynamically distinct; i.e. they are not homeomorphic via straightening maps. More precisely, the corresponding straightening maps are discontinuous at all but possibly a discrete set of parameters on certain sub-arcs (of suitable parabolic arcs).*

In general, we conjecture that there is no ‘universal’ formula for $\text{ind}_{\mathcal{C}}$.

Conjecture 9.3 (Baby Multicorns are Dynamically Distinct - Strong Version). *Let \mathcal{C}_1 and \mathcal{C}_2 be two distinct parabolic arcs in \mathcal{M}_d^* such that $\omega^i \mathcal{C}_1 \neq \mathcal{C}_2$ for $i = 1, 2, \dots, d+1$. Then the functions $\text{ind}_{\mathcal{C}_1}$ and $\text{ind}_{\mathcal{C}_2}$ are not identically equal. Therefore two multicorn-like sets, which are not ω^i -rotates of each other, are dynamically distinct; i.e. they are not homeomorphic via straightening maps.*

Remark. 1) Let us define a map $\xi : \overline{\mathcal{C} \cap \partial H'} \rightarrow \overline{\chi(\mathcal{C}) \cap \partial \chi(H')}$ by sending the parabolic cusp (on $\overline{\mathcal{C} \cap \partial H'}$) to the parabolic cusp (on $\overline{\chi(\mathcal{C}) \cap \partial \chi(H')}$), and sending the unique parameter (on $\mathcal{C} \cap \partial H'$) with parabolic fixed point index τ to the unique parameter (on $\chi(\mathcal{C}) \cap \partial \chi(H')$) with parabolic fixed point index τ . This definition makes sense because $\text{ind}_{\mathcal{C}}(\mathcal{C} \cap \partial H') = [1, +\infty)$ (respectively $\text{ind}_{\chi(\mathcal{C})}(\chi(\mathcal{C}) \cap \partial \chi(H')) = [1, +\infty)$), and $\text{ind}_{\mathcal{C}}$ (respectively $\text{ind}_{\chi(\mathcal{C})}$) is strictly increasing (in particular, injective) there (see Corollary 2.10).

Using Lemma 2.9, it is not hard to see that ξ is a continuous extension of $\chi|_{H'}$ to $\overline{\mathcal{C} \cap \partial H'}$. Therefore, Conjecture 9.3 implies that the maps ξ and χ do not agree everywhere on $\mathcal{C} \cap \partial H'$.

2) Note that the original multicorn \mathcal{M}_d^* has a $(d+1)$ -fold rotational symmetry (compare Lemma 2.2). Conjecture 9.3 implies that the multicorn-like sets do not have any such symmetry.

10. RECOVERING PARABOLIC MAPS FROM THEIR GERMS

Recall that the one of the key steps in the proof of Theorem 1.2 was to extend a carefully constructed local (germ) conjugacy to a semi-local (polynomial-like map) conjugacy, which allowed us to conclude that the corresponding polynomials are affinely conjugate. The extension of the germ conjugacy (see Lemma 4.2) made use of some of its special properties; in particular, we used the fact that the germ conjugacy preserves the post-critical orbits. However, in general, a conjugacy between two polynomial parabolic germs has no reason to preserve the post-critical orbits (germ conjugacies are defined locally, and post-critical orbits are global objects). Motivated by this, we asked some general local-global questions about polynomial parabolic germs in Question 3.4.

In this section, we will prove a rigidity property of unicritical holomorphic and antiholomorphic parabolic polynomials. More precisely, we will show that a unicritical holomorphic polynomial having a parabolic cycle is completely determined by the conformal conjugacy class of its parabolic germ or equivalently, by its Ecalle-Voronin invariants.

We will need the concept of extended horn maps, which are the natural maximal extensions of the horn maps. For the sake of completeness, we include the basic definitions, and properties of horn maps. For simplicity, we will only define it in the context of parabolic points with multiplier 1, and a single petal. More comprehensive accounts on these ideas can be found in [BE02, §2].

Let p be a (parabolic) holomorphic polynomial, z_0 be such that $p^{\circ k}(z_0) = z_0$, and $p^{\circ k}(z) = z + (z - z_0)^2 + \mathcal{O}(|z - z_0|^3)$ locally near z_0 . The parabolic point z_0 of p has exactly two petals, one attracting and one repelling (denoted by \mathcal{P}^{att} and \mathcal{P}^{rep} respectively). The intersection of the two petals has two connected components. We denote by \mathcal{U}^+ the connected component of $\mathcal{P}^{\text{att}} \cap \mathcal{P}^{\text{rep}}$ whose image under the Fatou coordinates is contained in the upper half-plane, and by \mathcal{U}^- the one whose image under the Fatou coordinates is contained in the lower half-plane. We define the “sepals” \mathcal{S}^\pm by

$$\mathcal{S}^\pm = \bigcup_{n \in \mathbb{Z}} p^{\circ n}(\mathcal{U}^\pm)$$

Note that each sepal contains a connected component of the intersection of the attracting and the repelling petals, and they are invariant under the first holomorphic return map of the parabolic point. The attracting Fatou coordinate ψ^{att} (respectively the repelling Fatou coordinate ψ^{rep}) can be extended to $\mathcal{P}^{\text{att}} \cup \mathcal{S}^+ \cup \mathcal{S}^-$ (respectively to $\mathcal{P}^{\text{rep}} \cup \mathcal{S}^+ \cup \mathcal{S}^-$) such that they conjugate the first holomorphic return map to the translation $\zeta \mapsto \zeta + 1$.

Definition (Lifted horn maps). Let us define $V^- = \psi^{\text{rep}}(\mathcal{S}^-)$, $V^+ = \psi^{\text{rep}}(\mathcal{S}^+)$, $W^- = \psi^{\text{att}}(\mathcal{S}^-)$, and $W^+ = \psi^{\text{att}}(\mathcal{S}^+)$. Then, denote by $H^- : V^- \rightarrow W^-$ the restriction of $\psi^{\text{att}} \circ (\psi^{\text{rep}})^{-1}$ to V^- , and by $H^+ : V^+ \rightarrow W^+$ the restriction of $\psi^{\text{att}} \circ (\psi^{\text{rep}})^{-1}$ to V^+ . We refer to H^\pm as lifted horn maps for p at z_0 .

Lifted horn maps are unique up to pre- and post-composition by translation. Note that such translations must be composed with both of the H^\pm at the same time. The regions V^\pm and W^\pm are invariant under translation by 1. Moreover, the asymptotic development of the Fatou coordinates implies that the regions V^+ and W^+ contain an upper half-plane, whereas the regions V^- and W^- contain a lower half-plane. Consequently, under the projection $\Pi : \zeta \mapsto w = \exp(2i\pi\zeta)$, the regions V^+ and W^+ project to punctured neighborhoods \mathcal{V}^+ and \mathcal{W}^+ of 0, whereas V^- and W^- project to punctured neighborhoods \mathcal{V}^- and \mathcal{W}^- of ∞ .

The lifted horn maps H^\pm satisfy $H^\pm(\zeta + 1) = H^\pm(\zeta) + 1$ on V^\pm . Thus, they project to mappings $h^\pm : \mathcal{V}^\pm \rightarrow \mathcal{W}^\pm$ such that the following diagram commutes:

$$\begin{array}{ccc} V^\pm & \xrightarrow{H^\pm} & W^\pm \\ \downarrow \Pi & & \downarrow \Pi \\ \mathcal{V}^\pm & \xrightarrow{h^\pm} & \mathcal{W}^\pm \end{array}$$

It is well-known that $\exists \eta, \eta' \in \mathbb{C}$ such that $H^+(\zeta) \approx \zeta + \eta$ when $\Im(\zeta) \rightarrow +\infty$, and $H^-(\zeta) \approx \zeta + \eta'$ when $\Im(\zeta) \rightarrow -\infty$. This proves that $h^+(w) \rightarrow 0$ as $w \rightarrow 0$. Thus, h^+ extends analytically to 0 by $h^+(0) = 0$. One can show similarly that h^- extends analytically to ∞ by $h^-(\infty) = \infty$.

Definition (Horn Maps). The maps $h^+ : \mathcal{V}^+ \cup \{0\} \rightarrow \mathcal{W}^+ \cup \{0\}$, and $h^- : \mathcal{V}^- \cup \{\infty\} \rightarrow \mathcal{W}^- \cup \{\infty\}$ are called *horn maps* for p at z_0 .

Let U_0 be the immediate basin of attraction of z_0 . Then there exists an extended attracting Fatou coordinate $\psi^{\text{att}} : U_0 \rightarrow \mathbb{C}$ (which is a ramified covering ramified only over the pre-critical points of $p^{\circ k}$ in U_0) satisfying $\psi^{\text{att}}(p^{\circ k}(z)) = \psi^{\text{att}}(z) + 1$, for every $z \in U_0$ (compare Figure 13). Similarly, the inverse of the repelling Fatou coordinate ψ^{rep} at z_0 extends to a holomorphic map $\zeta^{\text{rep}} : \mathbb{C} \rightarrow \mathbb{C}$ satisfying $p^{\circ k}(\zeta^{\text{rep}}(\zeta)) = \zeta^{\text{rep}}(\zeta + 1)$, for every $\zeta \in \mathbb{C}$. We define D_0^+ (respectively D_0^-) to be the connected component of $(\zeta^{\text{rep}})^{-1}(U_0)$ containing an upper half plane (respectively a lower half plane). Furthermore, let \mathcal{D}_0^\pm be the image of D_0^\pm under the projection $\Pi : \zeta \mapsto w = \exp(2i\pi\zeta)$.

Definition (Extended Horn Map). The maps $H^\pm := \psi^{\text{att}} \circ \zeta^{\text{rep}} : D_0^\pm \rightarrow \mathbb{C}$ are called the *extended lifted horn maps* for p at z_0 . They project (under Π) to the holomorphic maps $h^\pm : \mathcal{D}_0^\pm \rightarrow \hat{\mathbb{C}}$, which are called the *extended horn maps* for p at z_0 .

We will mostly work with the horn map $h^+ : \mathcal{D}_0^+ \rightarrow \hat{\mathbb{C}}$. Note that $\mathcal{D}_0^+ \cup \{0\}$ is the maximal domain of analyticity of the map h^+ . This can be seen as follows (see [LY14, Theorem 2.31] for a more general assertion of this type). Let $z' \in \partial U_0$, then there exists a sequence of pre-parabolic points $\{z_n\}_{n \geq 1} \subset \partial U_0$ converging to z' such that for each n , there is an arc $\gamma_n : (0, 1] \rightarrow U_0$ with $\gamma_n(0) = z_n$ satisfying the properties $\Re(\psi^{\text{att}}(\gamma_n(0, 1])) = \text{constant}$, and $\lim_{s \downarrow 0} \Im(\psi^{\text{att}}(\gamma_n(s))) = +\infty$. Therefore, for every $w' \in \partial \mathcal{D}_0^+$,

there exists a sequence of points $\{w_n\}_{n \geq 1} \subset \partial \mathcal{D}_0^+$ converging to w' such that for each n , there is an arc $\Gamma_n : [0, 1] \rightarrow \mathcal{D}_0^+$ with $\Gamma_n(0) = w_n$ satisfying $\lim_{s \downarrow 0} h^+(\Gamma_n(s)) = 0$. It follows from the identity principle that if we could continue h^+ analytically in a neighborhood of w' , then h^+ would be identically 0, which is a contradiction to the fact that h^+ is asymptotically a rotation near 0.

With these preparations, we are now ready to prove our first local-global principle for parabolic germs.

Definition.

- Let $\mathcal{M}_d^{\text{cusp}}$ be the union of the set of all root points of the primitive hyperbolic components, and the set of all co-root points of the multibrot set \mathcal{M}_d (compare [EMS16, §3.2]. For $c_1, c_2 \in \mathcal{M}_d^{\text{cusp}}$, we write $c_1 \sim c_2$ if $z^d + c_1$ and $z^d + c_2$ are affinely conjugate; i.e. if c_2/c_1 is a $(d-1)$ -th root of unity. We denote the set of equivalence classes under this equivalence relation by $(\mathcal{M}_d^{\text{cusp}}/\sim)$.
- Let $\text{Diff}^{+1}(\mathbb{C}, 0)$ be the set of conformal conjugacy classes of holomorphic germs (at 0) fixing 0, and having multiplier $+1$ at 0.

For $c \in \mathcal{M}_d^{\text{cusp}}$, let z_c be the characteristic parabolic point of $p_c(z) = z^d + c$, and k be the period of z_c . Conjugating $p_c^{\circ k}|_{N_{z_c}}$ (where N_{z_c} is a sufficiently

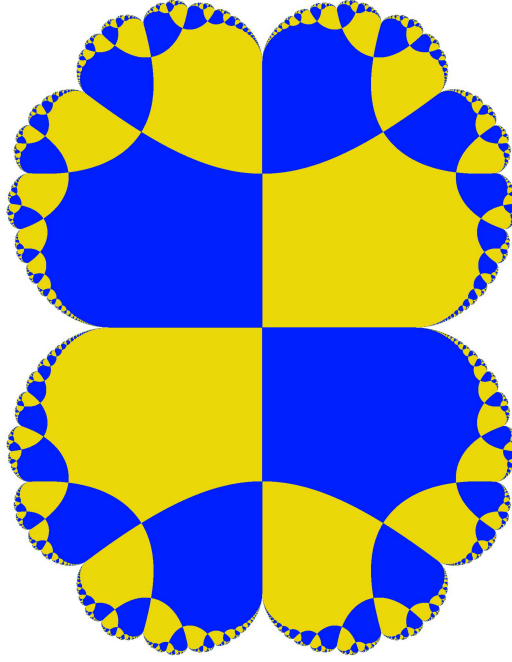


FIGURE 13. The parabolic chessboard for the polynomial $z + z^2$: normalizing $\psi^{\text{att}}(-\frac{1}{2}) = 0$, each yellow tile biholomorphically maps to the upper half plane, and each blue tile biholomorphically maps to the lower half plane under ψ^{att} . The pre-critical points of $z + z^2$ or equivalently the critical points of ψ^{att} are located where four tiles meet (Figure Courtesy Arnaud Chéritat).

small neighborhood of z_c) by an affine map that sends z_c to the origin, one obtains an element of $\text{Diff}^{+1}(\mathbb{C}, 0)$. The following lemma settles the germ rigidity for parameters in $\mathcal{M}_d^{\text{cusp}}$ (i.e. for parabolic parameters with a single petal).

Lemma 10.1 (Parabolic Germs Determine Co-roots, and Roots of Primitive Components). *The map*

$$\bigsqcup_{d \geq 2} (\mathcal{M}_d^{\text{cusp}} / \sim) \rightarrow \text{Diff}^{+1}(\mathbb{C}, 0)$$

$$c \mapsto p_c^{\text{ok}}|_{N_{z_c}}$$

is injective.

Proof. For $i = 1, 2$, let $c_i \in \mathcal{M}_{d_i}^{\text{cusp}}$, the parabolic cycle of c_i have period k_i , the characteristic parabolic points of $p_{c_i}(z) = z^{d_i} + c_i$ be z_i , and the characteristic Fatou components of p_{c_i} be U_i .

We assume that $g_1 := p_{c_1}^{\circ k_1}|_{N_{z_1}}$ and $g_2 := p_{c_2}^{\circ k_2}|_{N_{z_2}}$ are conformally conjugate by some local biholomorphism $\varphi : N_{z_1} \rightarrow N_{z_2}$. Then these two germs have the same horn map germ at 0, and hence p_{c_1} and p_{c_2} have the same extended horn map h^+ (recall that the domain of h^+ is its maximal domain of analyticity; i.e. h^+ is completely determined by the germ of the horn map at 0). If $\psi_{c_2}^{\text{att}}$ is an extended attracting Fatou coordinate for p_{c_2} at z_2 , then there exists an extended attracting Fatou coordinate $\psi_{c_1}^{\text{att}}$ for p_{c_1} at z_1 such that $\psi_{c_1}^{\text{att}} = \psi_{c_2}^{\text{att}} \circ \varphi$ in their common domain of definition. By [BE02, Proposition 4], h^+ is a ramified covering with the unique critical value $\Pi(\psi_{c_1}^{\text{att}}(c_1)) = \Pi(\psi_{c_2}^{\text{att}}(c_2))$. Note that the ramification index of h^+ over this unique critical value is $d_1 - 1 = d_2 - 1$. This shows that $d_1 = d_2$.

Furthermore, $\psi_{c_1}^{\text{att}}(c_1) - \psi_{c_2}^{\text{att}}(c_2) = n \in \mathbb{Z}$. We can normalize our attracting Fatou coordinates such that $\psi_{c_1}^{\text{att}}(c_1) = 0$, and $\psi_{c_2}^{\text{att}}(c_2) = -n$. Put $\eta := g_2^{\circ(-n)} \circ \varphi$. Then, η is a new conformal conjugacy between g_1 and g_2 . We stick to the Fatou coordinate $\psi_{c_1}^{\text{att}}$ for p_{c_1} , and define a new Fatou coordinate $\tilde{\psi}_{c_2}^{\text{att}}$ for p_{c_2} such that $\psi_{c_1}^{\text{att}} = \tilde{\psi}_{c_2}^{\text{att}} \circ \eta$ in their common domain of definition. Let N be large enough so that $p_{c_2}^{\circ k_2(N+n)}(c_2)$ is contained in the domain of definition of φ^{-1} . Now,

$$\begin{aligned} \tilde{\psi}_{c_2}^{\text{att}}(c_2) &= \tilde{\psi}_{c_2}^{\text{att}}(p_{c_2}^{\circ N k_2}(c_2)) - N \\ &= \psi_{c_1}^{\text{att}}\left(\varphi^{-1}\left(p_{c_2}^{\circ(N+n)k_2}(c_2)\right)\right) - N \\ &= \psi_{c_2}^{\text{att}}\left(p_{c_2}^{\circ(N+n)k_2}(c_2)\right) - N \\ &= \psi_{c_2}^{\text{att}}(c_2) + n + N - N \\ &= 0 \end{aligned}$$

Therefore, we have a germ conjugacy η such that the Fatou coordinates of p_{c_1} and p_{c_2} satisfy the following properties

$$\psi_{c_1}^{\text{att}} = \tilde{\psi}_{c_2}^{\text{att}} \circ \eta, \quad \psi_{c_1}^{\text{att}}(c_1) = 0, \quad \tilde{\psi}_{c_2}^{\text{att}}(c_2) = 0.$$

Since $p_{c_i}^{\circ k_i}|_{U_i}$ has a unique critical point of the same degree, one concludes (e.g. arguing as in Lemma 4.2) that η extends to a conformal conjugacy between $p_{c_1}^{\circ k_1}|_{U_1}$ and $p_{c_2}^{\circ k_2}|_{U_2}$. Now arguing as in Lemma 4.3, one sees that η extends to a conformal conjugacy between some neighborhoods of $\overline{U_i}$. The condition that c_i is a root point of a primitive hyperbolic component or a co-root point implies that z_i has exactly one attracting petal, and hence η induces a conformal conjugacy between the polynomial-like restrictions of $p_{c_1}^{\circ k_1}$ and $p_{c_2}^{\circ k_2}$ in some neighborhoods of $\overline{U_1}$ and $\overline{U_2}$ respectively.

We can now invoke [Ino11, Theorem 1] to deduce the existence of polynomials h , h_1 and h_2 such that $p_{c_1}^{\circ k_1} \circ h_1 = h_1 \circ h$, $p_{c_2}^{\circ k_2} \circ h_2 = h_2 \circ h$. In particular, we have that $\deg(p_{c_1}^{\circ k_1}) = d_1^{k_1} = d_1^{k_2} = \deg(p_{c_2}^{\circ k_2})$. Hence, $k_1 = k_2$.

The rest of the proof is similar to the reduction step of Ritt and Engstrom as in the proof of Theorem 1.2, so we only give a sketch. If both h_1 and

h_2 are of degree 1, then we are done. If $\deg(h_i) > 1$ for some i , then using Theorem 4.6, and the fact that $p_{c_i}^{\circ k_i}$ has no finite critical orbit, we conclude that $\gcd(\deg(p_{c_i}^{\circ k_i}), \deg(h_i)) > 1$. Now applying Engstrom's theorem [Eng41] (compare [Ino11, Theorem 11, Corollary 12, Lemma 13]), we obtain the existence of polynomials (of degree at least two) α_i, β_i such that up to affine conjugacy,

$$(3) \quad p_{c_i}^{\circ k_i} = \alpha_i \circ \beta_i \text{ and } h = \beta_i \circ \alpha_i$$

Either $\deg(h_i) > 1$ for $i = 1$ and 2, or one of the h_i is an affine conjugacy. In either case, by looking at the multiplicities of the critical points of h , and relation (3), we can deduce that $c_1 = c_2$ (up to affine conjugacy). \square

Here is an interesting corollary of the proof of the previous theorem.

Corollary 10.2 (Injectivity of Unicritical Renormalization Operator). *Let $c_1, c_2 \in \mathcal{M}_d$ have polynomial-like restrictions (renormalizations) $\mathcal{R}p_{c_1} := p_{c_1}^{\circ k_1} : U_1 \rightarrow V_1$, and $\mathcal{R}p_{c_2} := p_{c_2}^{\circ k_2} : U_2 \rightarrow V_2$ such that $\mathcal{R}p_{c_1}$ is not post-critically finite. If $\mathcal{R}p_{c_1}$ and $\mathcal{R}p_{c_2}$ are holomorphically conjugate, then p_{c_1} and p_{c_2} are affinely conjugate.*

Using essentially the same ideas, one can prove a variant of the above result for polynomials of arbitrary degree, provided that the parabolic point has exactly one petal, and its immediate basin of attraction contains exactly one critical point (of possibly higher multiplicity).

Proposition 10.3 (Unicritical Basins). *Let p_1 and p_2 be two polynomials (of any degree) satisfying $p_i(0) = 0$, and $p_i(z) = z + z^2 + \mathcal{O}(|z|^3)$ locally near 0. Let U_i be the immediate basin of attraction of p_i at 0, and p_i has exactly one critical point of multiplicity k_i in U_i . If p_1 and p_2 are (locally) conformally conjugate in some neighborhoods of 0, then $k_1 = k_2$, and there exist polynomials h, h_1 and h_2 such that $p_1 \circ h_1 = h_1 \circ h$, $p_2 \circ h_2 = h_2 \circ h$. In particular, $\deg(p_1) = \deg(p_2)$.*

Let us now proceed to the proof of Theorem 1.4.

Proof of Theorem 1.4. The number of petals of a parabolic germ is a topological conjugacy invariant. If the parabolic cycles of the polynomials $z^{d_1} + c_1$ and $z^{d_2} + c_2$ have a single petal, then we are in the case of Lemma 10.1, and we have $d_1 = d_2$ and $c_1 = c_2$.

Henceforth, we assume that c_1 and c_2 are roots of some satellite components of \mathcal{M}_{d_1} and \mathcal{M}_{d_2} respectively. Let the period of the parabolic cycle of p_{c_i} be k_i (so c_i sits on the boundary of a hyperbolic component of period k_i and a hyperbolic component of period n_i). Set $q_i := n_i/k_i$. Then it is easy to verify that we have $p_{c_i}^{\circ k_i}(z) = e^{\frac{2\pi p_i i}{q_i}} z + b_i(z - z_i)^{r_i+1} + \mathcal{O}((z - z_i)^{r_i+2})$ for an integer $r_i \geq 1$, $p_i \in \mathbb{Z}/q_i\mathbb{Z}$, and $b_i \in \mathbb{C}$, as the Taylor expansion of $p_{c_i}^{\circ k_i}$ near z_i . If the parabolic germs of $p_{c_i}^{\circ k_i}$ (for $i = 1, 2$) are conformally conjugate, then we have $p := p_1 = p_2$, and $q := q_1 = q_2 > 1$. Here q is the number of attracting petals at the parabolic point z_i , and these petals are

permuted transitively by $p_{c_i}^{\circ k_i}$. Furthermore, by looking at the ramification index of the unique singular value of their common horn maps, we deduce that $d_1 = d_2$. We will denote this common degree by d .

Since c_i is the root of a satellite component attached to some hyperbolic component of period k_i , the polynomial $p_{c_i}^{\circ k_i}$ has a polynomial-like restriction h_i that is hybrid equivalent to some (degree d) “fat” $\frac{p}{q}$ -rabbit (basilica if $q = 2$) parameter on the boundary of the principal hyperbolic component of \mathcal{M}_d (more precisely, $f_{c_i}^{\circ k_i}$ has a polynomial-like restriction h_i that is hybrid equivalent to some parameter c'_i such that $f_{c'_i}$ has a fixed point of multiplier $e^{\frac{2\pi i p}{q}}$). Arguments as in the proof of Lemma 4.2 involving computations with Fatou coordinates and Riemann maps of the immediate basins of attraction of the characteristic parabolic point of f_{c_i} show that the local germ conjugacy can be extended to the union of the q periodic Fatou components of $K(h_i)$. Subsequently, as in Lemma 4.2, one can use the dynamics and the functional equation to obtain a holomorphic conjugacy between the polynomial-like maps h_1 and h_2 . It then follows from Corollary 10.2 that $c_1 = c_2$ up to affine conjugacy. \square

The fundamental factor that makes the above proofs work is unicriticality since one can read off the conformal position of the unique critical value from the extended horn map. The next best family of polynomials, where this philosophy can be applied, is $\{f_c^{\circ 2}\}_{c \in \mathbb{C}}$. The proof of rigidity of parabolic parameters of \mathcal{M}_d^* comes in two different flavors. The fact that the even period parabolic parameters of \mathcal{M}_d^* are completely determined (up to affine conjugacy) by their parabolic germs follows by an argument similar to the one employed in the proof of Theorem 1.4. The case of odd period non-cusp parabolic parameters are, however, slightly more tricky.

We define $\Omega_d^{\text{odd}} := \{c \in \mathbb{C} : f_c(z) = \bar{z}^d + c \text{ has a parabolic cycle of odd period with a single petal}\}$, and $\Omega_d^{\text{even}} := \{c \in \mathbb{C} : f_c(z) = \bar{z}^d + c \text{ has a parabolic cycle of even period}\}$. For $c_1, c_2 \in \Omega_d^{\text{odd}} \cup \Omega_d^{\text{even}}$, we write $c_1 \sim c_2$ if $\bar{z}^d + c_1$ and $\bar{z}^d + c_2$ are affinely conjugate; i.e. if c_2/c_1 is a $(d+1)$ -th root of unity. We denote the set of equivalence classes under this equivalence relation by $\Omega_d^{\text{odd}} \cup \Omega_d^{\text{even}} / \sim$. By abusing notation, we will identify c_i with its equivalence class in $\Omega_d^{\text{odd}} \cup \Omega_d^{\text{even}} / \sim$. The first obstruction to recovering f_c from its parabolic germ comes from the following observation: if $c \in \Omega_d^{\text{odd}}$ has a parabolic cycle of odd period k , then the characteristic parabolic germs of $f_c^{\circ 2k}$ and $f_{c^*}^{\circ 2k}$ are conformally conjugate by the map $\iota \circ f_c^{\circ k}$. The next theorem will show that this is, in fact, the only obstruction.

Theorem 10.4 (Recovering Anti-polynomials from Their Parabolic Germs). *For $i = 1, 2$, let $c_i \in (\Omega_{d_i}^{\text{odd}} \cup \Omega_{d_i}^{\text{even}} / \sim)$, z_i be the characteristic parabolic point of f_{c_i} , and k_i be the period of z_i under $f_{c_i}^{\circ 2}$. If the parabolic germs $f_{c_1}^{\circ 2k_1}$ and $f_{c_2}^{\circ 2k_2}$ around z_1 and z_2 (respectively) are conformally conjugate, then $d_1 = d_2 = d$ (say), and one of the following is true.*

- (1) $c_1, c_2 \in \Omega_d^{\text{even}}$, and $c_1 = c_2$ in $\Omega_d^{\text{even}}/\sim$.
- (2) $c_1, c_2 \in \Omega_d^{\text{odd}}$, and $c_2 \in \{c_1, c_1^*\}$ in $\Omega_d^{\text{odd}}/\sim$.

Proof. Let U_i be the characteristic Fatou component of f_{c_i} . Note that by [BE02, Proposition 4], if $c_1 \in \Omega_{d_1}^{\text{even}}$, then the corresponding (upper) extended horn map(s) has (have) exactly one singular value. On the other hand, if $c_1 \in \Omega_{d_1}^{\text{odd}}$, then the corresponding (upper) extended horn map(s) has (have) exactly two distinct singular values. Since the parabolic germs of $f_{c_1}^{\circ 2k_1}$ and $f_{c_2}^{\circ 2k_2}$ are conformally conjugate, they have common (upper) extended horn map(s). By looking at the number of singular values of the common extended horn map(s), and their ramification indices, we conclude that

- i) $d_1 = d_2 = d$ (say), and
- ii) either both the c_i are in Ω_d^{odd} , or both the c_i are in Ω_d^{even} .

If both c_i are in Ω_d^{even} , then the first holomorphic return maps of U_i are conformally conjugate (in fact, they are conjugate to the same Blaschke product on \mathbb{D}). Therefore arguments similar to the ones employed in the proof of Theorem 1.4 show that f_{c_1} and f_{c_2} are affinely conjugate. Hence, $c_2 = c_1$ in $\Omega_d^{\text{even}}/\sim$.

The case when both c_i are in Ω_d^{odd} is more delicate because the conformal conjugacy class of $f_{c_i}^{2k_i}|_{U_i}$ depends on the critical Ecalle height of f_{c_i} . We assume that $g_1 := f_{c_1}^{\circ 2k_1}|_{N_{z_1}}$ and $g_2 := f_{c_2}^{\circ 2k_2}|_{N_{z_2}}$ are conformally conjugate by some local biholomorphism $\varphi : N_{z_1} \rightarrow N_{z_2}$ (where N_{z_i} is a sufficiently small neighborhood of z_i). Then these two germs have the same horn map germ at 0, and hence $f_{c_1}^{\circ 2k_1}$ and $f_{c_2}^{\circ 2k_2}$ have the same extended horn map h^+ at 0 (recall that the domain of h^+ is its maximal domain of analyticity; i.e. h^+ is completely determined by the germ of the horn map at 0). If $\psi_{c_2}^{\text{att}}$ is an extended attracting Fatou coordinate for f_{c_2} at z_2 (normalized so that the attracting equator maps to the real line), then there exists an extended attracting Fatou coordinate $\psi_{c_1}^{\text{att}}$ for f_{c_1} at z_1 such that $\psi_{c_1}^{\text{att}} = \psi_{c_2}^{\text{att}} \circ \varphi$ in their common domain of definition. Since any local conformal conjugacy preserves equators, $\psi_{c_1}^{\text{att}}$ maps the attracting equator of f_{c_1} at z_1 to the real line. By [BE02, Proposition 4], h^+ is a ramified covering with exactly two critical values. This implies that

$$\{\Pi(\psi_{c_1}^{\text{att}}(c_1)), \Pi(\psi_{c_1}^{\text{att}}(f_{c_1}^{\circ k_1}(c_1)))\} = \{\Pi(\psi_{c_2}^{\text{att}}(c_2)), \Pi(\psi_{c_2}^{\text{att}}(f_{c_2}^{\circ k_2}(c_2)))\}.$$

We now consider two cases.

Case 1: $\Pi(\psi_{c_1}^{\text{att}}(c_1)) = \Pi(\psi_{c_2}^{\text{att}}(c_2))$. We can assume, possibly after adjusting the horizontal degree of freedom of the Fatou coordinates, and modifying the conformal conjugacy φ (as in the proof of Theorem 1.4) that

$$\psi_{c_1}^{\text{att}} = \psi_{c_2}^{\text{att}} \circ \varphi, \quad \psi_{c_1}^{\text{att}}(c_1) = it = \psi_{c_2}^{\text{att}}(c_2).$$

In particular, f_{c_1} and f_{c_2} have equal critical Ecalle height, and hence $f_{c_1}^{\circ 2k_1}|_{U_1}$ and $f_{c_2}^{\circ 2k_2}|_{U_2}$ are conformally conjugate. Arguing as in Lemma 4.2,

and Lemma 4.3, we see that φ extends to a conformal conjugacy between $f_{c_1}^{\circ 2k_1}$ and $f_{c_2}^{\circ 2k_2}$ restricted to some neighborhoods of $\overline{U_i}$. Since c_i is a non-cusp parameter, z_i has exactly one attracting petal, and hence φ induces a conformal conjugacy between the polynomial-like restrictions of $f_{c_1}^{\circ 2k_1}$ and $f_{c_2}^{\circ 2k_2}$ in some neighborhoods of $\overline{U_1}$ and $\overline{U_2}$ respectively.

We can now invoke [Ino11, Theorem 1] to deduce the existence of polynomials h , h_1 and h_2 such that $f_{c_1}^{\circ 2k_1} \circ h_1 = h_1 \circ h$, $f_{c_2}^{\circ 2k_2} \circ h_2 = h_2 \circ h$. In particular, we have that $d^{2k_1} = \deg(f_{c_1}^{\circ 2k_1}) = \deg(f_{c_2}^{\circ 2k_2}) = d^{2k_2}$. Hence, $k_1 = k_2$. Finally, applying Ritt and Engstrom's reduction steps (similar to the proof of Theorem 1.2), we can conclude from the semi-conjugacy relations that f_{c_1} and f_{c_2} are affinely conjugate.

Case 2: $\Pi(\psi_{c_1}^{\text{att}}(c_1)) = \Pi(\psi_{c_2}^{\text{att}}(f_{c_2}^{\circ k_2}(c_2)))$. Since φ is a conformal conjugacy between $f_{c_1}^{\circ 2k_1}|_{N_{z_1}}$ and $f_{c_2}^{\circ 2k_2}|_{N_{z_2}}$, $\tilde{\varphi} := \iota \circ f_{c_2}^{\circ k_2} \circ \varphi$ is a conformal conjugacy between the characteristic parabolic germs of $f_{c_1}^{\circ 2k_1}$ and $f_{c_2^*}^{\circ 2k_2}$. Let $\psi_{c_2^*}^{\text{att}}$ be an extended attracting Fatou coordinate for $f_{c_2^*}$ at z_2^* such that $\psi_{c_2^*}^{\text{att}} \circ \iota \circ f_{c_2}^{\circ k_2} = \psi_{c_2}^{\text{att}}$ in their common domain of definition. Therefore,

$$\begin{aligned} \psi_{c_1}^{\text{att}} &= \psi_{c_2}^{\text{att}} \circ \varphi \\ &= \psi_{c_2^*}^{\text{att}} \circ \iota \circ f_{c_2}^{\circ k_2} \circ \varphi \\ &= \psi_{c_2^*}^{\text{att}} \circ \tilde{\varphi} \end{aligned}$$

in their common domain of definition.

Moreover, a simple computation shows that

$$\Pi(\psi_{c_1}^{\text{att}}(c_1)) = \Pi(\psi_{c_2^*}^{\text{att}}(c_2^*)).$$

The situation now reduces to that of Case (1), and a similar argumentation shows that f_{c_1} and $f_{c_2^*}$ are affinely conjugate.

Combining Case 1 and Case 2, we conclude that $c_2 \in \{c_1, c_1^*\}$ in $\Omega_d^{\text{odd}}/\sim$. \square

11. POLYNOMIALS WITH REAL-SYMMETRIC PARABOLIC GERMS

In this section, we will discuss another local-global principle for parabolic germs that are obtained by restricting a polynomial map of the plane near a parabolic fixed/periodic point. We say that a parabolic germ g at 0 is *real-symmetric* if in some conformal coordinates, $\overline{g(\bar{z})} = g(z)$; i.e. if all the coefficients in its power series expansion are real after a local conformal change of coordinates. This is a strong local condition, and we believe that in general, a polynomial parabolic germ can be real-symmetric only if the polynomial itself has a global antiholomorphic involutive symmetry.

Our proof of Theorem 1.2 shows that if $f_c(z) = \bar{z}^d + c$ has a simple (exactly one attracting petal) parabolic orbit of odd period, and if the critical Ecalle height is 0, then the corresponding parabolic germ is real-symmetric if and only if f_c commutes with a global antiholomorphic involution (see

Corollary 4.8). In this section, we generalize this result, and also prove the corresponding theorem for unicritical holomorphic polynomials.

We will make use of our discussion on extended horn maps in Section 10. The following characterization of real-symmetric parabolic germs, and the symmetry of its upper and lower horn maps will be useful for us. The result is classical [Lor06, §2.8.4].

Lemma 11.1. *For a simple parabolic germ g , the following are equivalent:*

- g is a real-symmetric germ,
- There is a g -invariant real-analytic curve Γ passing through the parabolic fixed point of g ,
- There is an antiholomorphic involution $\tilde{\iota}$ defined in a neighborhood of the parabolic fixed point (and fixing it) of g such that g commutes with $\tilde{\iota}$.

If any of these equivalent conditions are satisfied, one can choose attracting and repelling Fatou coordinates for g such that the involution $w \mapsto 1/\bar{w}$ is a conjugacy between the upper and lower horn map germs h^+ and h^- ; i.e. $1/\overline{h^-(1/\bar{w})} = h^+(w)$ for w near 0.

In fact, the statements about the horn map germs h^\pm (near 0 and ∞ respectively) can be made somewhat more global.

Lemma 11.2 (Extended Horn Maps for Real-symmetric Germs). *Let p be a polynomial with a simple parabolic fixed point z_0 such that the parabolic germ $p|_{B(z_0, \varepsilon)}$ (for some $\varepsilon > 0$ small enough) is real-symmetric. If we normalize the attracting and repelling Fatou coordinates of p at z_0 such that they map the real-analytic curve Γ to the real line, then the following is true for the corresponding horn maps: \mathcal{D}_0^- is the image of \mathcal{D}_0^+ under $w \mapsto 1/\bar{w}$, and $1/\overline{h^-(1/\bar{w})} = h^+(w)$ for all $w \in \mathcal{D}_0^+$.*

Proof. This follows from Lemma 11.1, and the fact that the horn map germs completely determine the extended horn maps h^\pm (since the extended horn maps are the maximal analytic continuations of the horn map germs). \square

Definition. We say that p_c (respectively f_c) is a real polynomial (respectively anti-polynomial) if p_c (respectively f_c) commutes with an antiholomorphic involution of the plane.

We now prove the following local-global principle for unicritical holomorphic polynomials. Recall that $\mathcal{M}_d^{\text{par}}$ is the set of all parabolic parameters of \mathcal{M}_d . For $c \in \mathcal{M}_d^{\text{par}}$, let z_c be the characteristic parabolic point, and U_c be the characteristic Fatou component (of period n) of $p_c(z) = z^d + c$.

Theorem 11.3 (Real-symmetric Germs Only for Real Polynomials). *Let c be in $\mathcal{M}_d^{\text{par}}$, z_c be the characteristic parabolic point of p_c , U_c be the characteristic Fatou component of p_c , and n be the period of the component U_c . If the parabolic germ $p_c^{\text{on}}|_{B(z_c, \varepsilon)}$ (for some $\varepsilon > 0$ small enough) is real-symmetric, then p_c is a real polynomial.*

Proof. We assume that the parabolic germ $g := p_c^{on}|_{B(z_c, \varepsilon)}$ is real-symmetric. Let α be a local conformal conjugacy between g , and a real germ h fixing 0. Observe that $\iota : z \mapsto z^*$ is an antiholomorphic conjugacy between p_c and p_{c^*} . It is easy to check that the germ $\iota \circ g \circ \iota = p_{c^*}^{on}|_{B(z_c^*, \varepsilon)}$ is also real-symmetric, and the local biholomorphism $\iota \circ \alpha \circ \iota$ conjugates the parabolic germ $\iota \circ g \circ \iota$ at z_c^* to the same real parabolic germ h as obtained above. Thus, the parabolic germs g at z_c , and $\iota \circ g \circ \iota$ at z_c^* are conformally conjugate by $\eta := (\iota \circ \alpha \circ \iota)^{-1} \circ \alpha$. Note that since the period of z_c and the period of U_c are not equal in general (they are different precisely when c is the root of a satellite component), we cannot apply Theorem 1.4 directly to this situation. However, there is a simple workaround. It follows from the construction that η respects the dynamics on the critical orbits. Hence η extends to the union of the periodic Fatou components touching at z_c (compare Lemma 4.2). Finally, we can analytically continue η to yield a conformal conjugacy between suitable polynomial-like restrictions of p_c^{on} and $p_{c^*}^{on}$ respectively. Now applying [Ino11, Theorem 1] and Ritt's classification of decomposition of polynomials (techniques that have been repeatedly used throughout this paper), we conclude that p_c and p_{c^*} are affinely conjugate. A straightforward computation shows that $c^* = \omega^j c$ where $\omega = \exp(\frac{2\pi i}{d-1})$, and $j \in \mathbb{N}$. But this precisely means that p_c commutes with the global antiholomorphic involution $\zeta \mapsto \omega^{-j} \zeta^*$. \square

Finally, let us record the analogue of Theorem 11.3 in the unicritical antiholomorphic family. The following theorem also sharpens Corollary 4.8. We continue with the terminologies introduced in the previous section.

Theorem 11.4 (Real-symmetric Germs Only for Real Anti-polynomials). *Let c be in $\Omega_d^{\text{odd}} \cup \Omega_d^{\text{even}}$, z_c be the characteristic parabolic point of f_c , U_c be the characteristic Fatou component of f_c , and n be the period of the component U_c . If the parabolic germ $f_c^{on}|_{B(z_c, \varepsilon)}$ (for some $\varepsilon > 0$ small enough) is real-symmetric, then f_c is a real anti-polynomial.*

Proof. The case when $c \in \Omega_{d_i}^{\text{even}}$ similar to the holomorphic case (Theorem 11.3). By a completely similar argument, we can conclude that $c^* = \omega^j c$ for some $j \in \{0, 1, \dots, d\}$, where $\omega = \exp(\frac{2\pi i}{d+1})$. But this is equivalent to saying that f_c commutes with the global antiholomorphic involution $\zeta \mapsto \omega^{-j} \zeta^*$.

Now we focus on the case $c \in \Omega_{d_i}^{\text{odd}}$. Note that in this case, the invariant real-analytic curve Γ passing through z_c (compare Lemma 11.1) is simply the union of the attracting equator at z_c , the parabolic point z_c , and the repelling equator at z_c . If we arrange our Fatou coordinates so that they map the equators to the real line, then the extended upper and lower horn maps of f_c^{on} at z_c are conjugated by $w \mapsto 1/\overline{w}$ (by Lemma 11.2). In particular, we have

$$\{\Pi(\psi_c^{\text{att}}(c)), \Pi(\psi_c^{\text{att}}(f_c^{on/2}(c)))\} = \{1/\overline{\Pi(\psi_c^{\text{att}}(c))}, 1/\overline{\Pi(\psi_c^{\text{att}}(f_c^{on/2}(c)))}\},$$

where ψ_c^{att} is our preferred attracting Fatou coordinate for f_c^{on} at z_c .

We can assume that $\psi_c^{\text{att}}(c) = it$, and $\psi_c^{\text{att}}(f_c^{\text{on}/2}(c)) = \frac{1}{2} - it$. Now a simple computation shows that we must have $\Pi(\psi_c^{\text{att}}(c)) = 1/\overline{\Pi(\psi_c^{\text{att}}(c))}$, and hence $t = 0$. Therefore, c is a critical Ecalle height 0 parameter.

Now as in the even period case, there exists a local conformal conjugacy α conjugating $f_c^{\text{on}}|_{B(z_c, \varepsilon)}$ to a real germ h fixing 0. Therefore, the local biholomorphism $\iota \circ \alpha \circ \iota$ conjugates the parabolic germ $f_{c^*}^{\text{on}}|_{B(z_{c^*}, \varepsilon)} = \iota \circ f_c^{\text{on}}|_{B(z_c, \varepsilon)} \circ \iota$ to the same real parabolic germ h as obtained above. It follows that $f_c^{\text{on}}|_{B(z_c, \varepsilon)}$ and $f_{c^*}^{\text{on}}|_{B(z_{c^*}, \varepsilon)}$ are conformally conjugate via $\eta := (\iota \circ \alpha \circ \iota)^{-1} \circ \alpha$, and η preserves the corresponding dynamically marked critical orbits (here we have used the fact that c is a critical Ecalle height 0 parameter). Choosing an extended attracting Fatou coordinate $\psi_{c^*}^{\text{att}}$ for f_{c^*} at z_{c^*} (normalized so that the attracting equator maps to the real line), we can find an extended attracting Fatou coordinate ψ_c^{att} for f_c at z_c such that $\psi_c^{\text{att}} = \psi_{c^*}^{\text{att}} \circ \eta$ in their common domain of definition. Moreover, by our construction of η , $\Pi(\psi_c^{\text{att}}(c)) = \Pi(\psi_{c^*}^{\text{att}}(c^*))$. It now follows from (Case 2 of the proof of) Theorem 10.4 that $c^* = \omega^j c$ for some $j \in \{0, 1, \dots, d\}$, where $\omega = \exp(\frac{2\pi i}{d+1})$. Therefore, f_c commutes with the global antiholomorphic involution $\zeta \mapsto \omega^{-j} \zeta^*$. \square

Remark. It follows from the proof of the above theorem that if an odd period non-cusp parabolic parameter of \mathcal{M}_d^* has a real-symmetric parabolic germ, then it must be a critical Ecalle height 0 parameter. This is another example where a global feature of the dynamics can be read off from its local properties.

REFERENCES

- [AKLS09] A. Avila, J. Kahn, M. Lyubich, and W. Shen. Combinatorial rigidity for unicritical polynomials. *Annals of Mathematics*, 170:783–797, 2009.
- [BBM] A. Bonifant, X. Buff, and J. Milnor. Antipode preserving cubic maps II: Tongues and the ring locus. Manuscript in preparation.
- [BBM15] A. Bonifant, X. Buff, and J. Milnor. Antipode preserving cubic maps: the fjord theorem. <http://www.math.univ-toulouse.fr/~buff/Preprints/Antipodal/Antipodal.pdf>, 2015.
- [BCL⁺15] A. Blokh, D. Childers, G. Levin, L. Oversteegen, and D. Schleicher. An extended Fatou-Shishikura inequality and wandering branch continua for polynomials. <http://arxiv.org/abs/1001.0953>, 2015.
- [BE02] X. Buff and A. L. Epstein. A parabolic Pommerenke-Levin-Yoccoz inequality. *Fund. Math.*, 172:249–289, 2002.
- [BE07] X. Buff and A. L. Epstein. From local to global analytic conjugacies. *Ergodic Theory and Dynamical Systems*, 27:1073–1094, 2007.
- [CFG15] J. Canela, N. Fagella, and A. Garijo. On a family of rational perturbations of the doubling map. <http://arxiv.org/abs/1501.03647>, 2015.
- [CHRSC89] W. D. Crowe, R. Hasson, P. J. Rippon, and P. E. D. Strain-Clark. On the structure of the Mandelbar set. *Nonlinearity*, 2, 1989.
- [DH85] A. Douady and J. H. Hubbard. On the dynamics of polynomial-like mappings. *Ann. Sci. Ec. Norm. Sup.*, 18:287–343, 1985.

- [Eca75] J. Ecalle. Théorie itérative : introduction à la théorie des invariants holomorphes. *J. Math. Pures Appl. (9)*, 54:183–25, 1975.
- [EMS16] D. Eberlein, S. Mukherjee, and D. Schleicher. Rational parameter rays of the multibrot sets. In *Dynamical Systems, Number Theory and Applications*, chapter 3, pages 49–84. World Scientific, 2016. http://dx.doi.org/10.1142/9789814699877_0003.
- [Eng41] H. T. Engstrom. Polynomial substitutions. *Amer. J. Math.*, 63:249–255, 1941.
- [GŚ97] J. Graczyk and G. Świątek. Generic hyperbolicity in the logistic family. *Annals of Mathematics*, 146:1–52, 1997.
- [HS14] J. H. Hubbard and D. Schleicher. Multicorns are not path connected. In *Frontiers in Complex Dynamics: In Celebration of John Milnor's 80th Birthday*, pages 73–102. Princeton University Press, 2014.
- [IK12] H. Inou and J. Kiwi. Combinatorics and topology of straightening maps, I: Compactness and bijectivity. *Advances in Mathematics*, 231:2666–2733, 2012.
- [IM16] H. Inou and S. Mukherjee. Non-landing parameter rays of the multicorns. *Inventiones Mathematicae*, 204:869–893, 2016. <http://dx.doi.org/10.1007/s00222-015-0627-3>.
- [Ino09] H. Inou. Combinatorics and topology of straightening maps II: Discontinuity. <http://arxiv.org/abs/0903.4289>, 2009.
- [Ino11] H. Inou. Extending local analytic conjugacies. *Trans. Amer. Math. Soc.*, 363:331–343, 2011.
- [Ino14] H. Inou. Self-similarity for the tricorn. <http://arxiv.org/pdf/1411.3081.pdf>, 2014.
- [Kaw07] T. Kawahira. A proof of simultaneous linearization with a polylog estimate. *Bulletin of the Polish Academy of Sciences (Mathematics)*, 55(1):43–52, 2007.
- [KN04] Y. Komori and S. Nakane. Landing property of stretching rays for real cubic polynomials. *Conformal Geometry and Dynamics*, 8:87–114, 2004.
- [KSS07] O. Kozlovski, W. Shen, and S. V. Strien. Density of hyperbolicity in dimension one. *Annals of Mathematics*, 166:145–182, 2007.
- [Lav89] P. Lavaurs. *Systèmes dynamiques holomorphes: explosion de points périodiques paraboliques*. PhD thesis, Université de Paris-Sud Centre d’Orsay, 1989.
- [Lor06] F. Loray. Pseudo-groupe d’une singularité de feuilletage holomorphe en dimension deux. <https://hal.archives-ouvertes.fr/hal-00016434/document>, 2006.
- [LY14] O. Lanford III and M. Yampolsky. *Fixed point of the parabolic renormalization operator*. Springer International Publishing, 1st edition, 2014.
- [Lyu99] M. Lyubich. Feigenbaum-Coullet-Tresser universality and Milnor’s hairiness conjecture. *Annals of Mathematics*, 149:319–420, 1999.
- [Mil92] J. Milnor. Remarks on iterated cubic maps. *Experiment. Math.*, 1:5–24, 1992.
- [Mil00] J. Milnor. On rational maps with two critical points. *Experiment. Math.*, 9:333–411, 2000.
- [Mil06] J. Milnor. *Dynamics in one complex variable*. Princeton University Press, New Jersey, 3rd edition, 2006.
- [MNS15] S. Mukherjee, S. Nakane, and D. Schleicher. On Multicorns and Unicorns II: bifurcations in spaces of antiholomorphic polynomials. *Ergodic Theory and Dynamical systems*, 2015. <http://dx.doi.org/10.1017/etds.2015.65>.
- [MP12] J. Milnor and A. Poirier. Hyperbolic components. <http://arxiv.org/abs/1205.2668>, 2012.
- [Muk15a] S. Mukherjee. *Antiholomorphic Dynamics: Topology of Parameter Spaces, and Discontinuity of Straightening*. PhD thesis, Jacobs University, Bremen, 2015.

- [Muk15b] S. Mukherjee. Orbit portraits of unicritical antiholomorphic polynomials. *Conformal Geometry and Dynamics of the AMS*, 19:35–50, 2015.
- [Muk15c] S. Mukherjee. Parabolic arcs of the multicorns: real-analyticity of hausdorff dimension, and singularities of $\text{Per}_n(1)$ curves. <http://arxiv.org/abs/1410.1180>, 2015.
- [Nak93] S. Nakane. Connectedness of the Tricorn. *Ergodic Theory and Dynamical Systems*, 13:349–356, 1993.
- [NS96] S. Nakane and D. Schleicher. The nonarcwise-connectedness of the tricorn. In *Problems concerning complex dynamical systems (Japanese)*, number 959 in *Sūrikaiseikikenkyūsho Kōkyūroku*, pages 73–83. Kyoto University, Research Institute for Mathematical Sciences, Kyoto, 1996.
- [NS03] S. Nakane and D. Schleicher. On Multicorns and Unicorns I : Antiholomorphic dynamics, hyperbolic components and real cubic polynomials. *International Journal of Bifurcation and Chaos*, 13:2825–2844, 2003.
- [PR08] C. L. Petersen and P. Roesch. The Yoccoz combinatorial analytic invariant. In *Holomorphic dynamics and renormalization: A Volume in Honour of John Milnor's 75th Birthday*, pages 145–176. AMS and The Fields Institute, 2008.
- [Rit22] J. F. Ritt. Prime and composite polynomials. *Trans. Amer. Math. Soc.*, 23:5166, 1922.
- [Sch99] D. Schleicher. On fibers and renormalization of Julia sets and Multibrot sets. <http://arxiv.org/pdf/math/9902156.pdf>, 1999.
- [Sch00] D. Schleicher. Rational parameter rays of the Mandelbrot set. *Astérisque*, 261:405–443, 2000.
- [Vor81] S. M. Voronin. Analytic classification of germs of conformal mappings $(\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$. *Funktsional. Anal. i Prilozhen*, 15:1–17, 1981.
- [Win90] R. Winters. *Bifurcations in families of antiholomorphic and bi-quadratic maps*. PhD thesis, Boston University, 1990.

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